AN ELEMENTARY COURSE

IN

ANALYTIC GEOMETRY

BY

J. H. TANNER

ASSISTANT PROFESSOR OF MATHEMATICS IN CORNELL UNIVERSITY

AND

JOSEPH ALLEN

FORMERLY INSTRUCTOR IN MATHEMATICS IN CORNELL UNIVERSITY
TUTOR IN THE COLLEGE OF THE CITY OF NEW YORK

INDIAN INSTITUTE OF SCIENCE
RECEIVED ON
20 DEC 1951
LIBRARY

NEW YORK ·:· CINCINNATI ·:· CHICAGO
AMERICAN BOOK COMPANY

13857
172

516.3
M98
[redacted]

[redacted]

Copyright, 1898, by
J. H. TANNER and JOSEPH ALLEN.

ANA. GEOM.

W. P. I

THE CORNELL MATHEMATICAL SERIES

LUCIEN AUGUSTUS WAIT . . . GENERAL EDITOR

(SENIOR PROFESSOR OF MATHEMATICS IN CORNELL UNIVERSITY)

THE CORNELL MATHEMATICAL SERIES.

LUCIEN AUGUSTUS WAIT,
(Senior Professor of Mathematics in Cornell University,)
GENERAL EDITOR.

This series is designed primarily to meet the needs of students in Engineering and Architecture in Cornell University; and accordingly many practical problems in illustration of the fundamental principles play an early and important part in each book.

While it has been the aim to present each subject in a simple manner, yet rigor of treatment has been regarded as more important than simplicity, and thus it is hoped that the series will be acceptable also to general students of Mathematics.

The general plan and many of the details of each book were discussed at meetings of the mathematical staff. A mimeographed edition of each volume was used for a term as the text-book in all classes, and the suggestions thus brought out were fully considered before the work was sent to press.

The series includes the following works:

ANALYTIC GEOMETRY. By J. H. Tanner and Joseph Allen.

DIFFERENTIAL CALCULUS. By James McMahon and Virgil Snyder.

INTEGRAL CALCULUS. By D. A. Murray.

PREFACE

ALTHOUGH in the writing of this book the needs of the students in the various departments of Engineering and of Architecture in Cornell University have received the first consideration, care has also been taken to make the work suitable for the general student and at the same time useful as an introduction to a more advanced course for those students who may wish to specialize later in mathematics.

Among the features of the book are:

(1) An extended introduction (Chaps. II, III, IV), in which it is hoped that the fundamental problems of the subject are clearly set forth and sufficiently illustrated. The chief difficulty which the beginner in Analytic Geometry usually has to overcome is the relation between an equation and its locus; having really mastered this, he easily and rapidly acquires a knowledge of the properties to which this relation leads, and especial care has therefore been given to this matter. Analytic Geometry is broader than Conic Sections, and it is the firm conviction of the authors that it is far more important to the student that he should acquire a familiarity with the spirit of the method of the subject than that he should be required to memorize the various properties of any particular curve.

(2) The making use of some intrinsic properties of curves (see Arts. 106, 112, 118), which experience with many classes has shown to give the student an unusually strong grasp on the equation of the second degree from which the xy-term is absent.

(3) Introduction of the demonstrations of general theorems by numerical examples. This not only makes clear to the student what is to be done, but shows also the method to be employed, — it *generalizes* after the student is acquainted with the *particular*.

(4) Easy but rigorous proofs of all the theorems within the scope of the book. *E.g.*, in Art. 67 it is proved, and

very simply, too, that the vanishing of the discriminant is not only a *necessary*, but also the *sufficient* condition that the quadratic equation represents a pair of straight lines.

It may also be mentioned here that, in the early part of the book, two or more figures are given in connection with a proof and so lettered that the same demonstration applies to each. It is hoped that this will help to convince the student of the generality of the demonstration. A copious index which enables one almost instantly to turn to anything contained in the book has also been added.

The engineering students at Cornell University study Analytic Geometry during the first term of their freshman year, and experience has shown that it is best to devote a few lessons at the beginning of the term to a rapid review of those parts of the Algebra and Trigonometry that are essential to the reading of the Analytic Geometry. The first twenty-three pages are devoted to this matter, and may, of course, be omitted by those classes that take up the subject immediately after reading the Algebra and Trigonometry.

The book contains little more than can be mastered by a properly prepared student of average ability in from twelve to fourteen weeks; if less than that time can be devoted to the work, the individual teacher will know best what parts may be most wisely omitted by his pupils. A list of lessons for a short course of eleven weeks is, however, suggested on the next two pages.

A few specific acknowledgments of indebtedness are made in foot-notes in the appropriate places in the book. Of the large number of examples which are inserted, many are original, while many others have come to be so common in text-books that no specific acknowledgment for them can be made. We take great pleasure in expressing here our thanks to the other authors of this series of books for their many helpful suggestions and criticisms; to our colleagues, Dr. J. I. Hutchinson and Dr. G. A. Miller, who have so greatly assisted us in reading the proof, and the latter of whom also read the manuscript before it went to press; to Mr. Peter Field, Fellow in Mathematics, and Mr. E. A. Miller for solving the entire list of examples; and to Mr. V. T. Wilson, Instructor in Drawing in Sibley College, for the care with which he has made the figures.

LIST OF LESSONS SUGGESTED FOR A SHORT COURSE

[From the various sets of exercises the teacher is expected to make selections for each lesson. The fifth day of each week should be devoted to reviewing the preceding four lessons.]

Lesson	Pages	Articles
1	1–9	1–8
2	9–15	9–12
3	15–23	13–17
4	24–28	18–22
5	29–33	23–27
6	34–40	28–30
7	40–42	31
8	43–52	32–37 As far as " Exercises," p. 52.
9	52–57	38–41
10	58–60	
11	61–65	42–45
12	65–73	46–48 With examples selected from p. 79.
13	73–80	49
14	81–85	50–53
15	86–94	54–58
16	94–98	59–61
17	98–104	62–63
18	105–110	64–65
19	110–115 118–119	66, 67, 69
20	119–122	
21	123–127 129–131	70–72 75–76

Lesson	Pages	Articles
22	131–137	77–78
23	137–142	79–82
24	142–149	83–85
25	149–155	86–90
26	156–165	93–100
27	165–169	
28	170–177	101–107
29	179–186	109–112
30	186–188 190–194	113, 115–117
31	195–202	118–122
32	203–208	123–126
33	209–216	127–132
34	216–218	
35	219–225	133–137
36	225–233	138–140
37	235–242	142–145
38	242–247 250–254	146–148 152–154
39	254–264	155–157
40	265–272	160–164
41	272–283	165–170
42	284–291	171–174
43	292–298	175–177
44	309–330	185–198

CONTENTS

PART I.—PLANE ANALYTIC GEOMETRY

CHAPTER I

INTRODUCTION

Algebraic and Trigonometric Conceptions

ARTICLE PAGE

1. Number 1
2. Constants and variables 2
3. Functions 3
4. Identity, equation, and root 4
5. Functions classified 4
6. Notation 5
7. Continuous and discontinuous functions 6
8., 9. The quadratic equation. Its solution 9
10. Zero and infinite roots 11
11. Properties of the quadratic equation 12
12. The quadratic equation involving two unknowns . . . 13

Trigonometric Conceptions and Formulas

13. Directed lines. Angles 15
14. Trigonometric ratios 17
15. Functions of related angles 18
16. Other important formulas 19
17. Orthogonal projection 21

CHAPTER II

GEOMETRIC CONCEPTIONS. THE POINT

I. *Coördinate Systems*

18. Coördinates of a point 24
19. Analytic Geometry 25

ARTICLE PAGE
20. Positive and negative coördinates 25
21. Cartesian coördinates of points in a plane 26
22. Rectangular coördinates 27
23. Polar coördinates 29
24. Notation 30

II. *Elementary Applications*

25. } Distance between two points
26. }
(1) Polar coördinates 31
(2) Cartesian coördinates; axes not rectangular . . . 32
(3) Rectangular coördinates 33
27. Slope of a line 33
28. Summary 34
29. The area of a triangle
(1) Rectangular coördinates 34
(2) Polar coördinates 36
30. To find the coördinates of the point which divides, in a given ratio, the straight line from one given point to another . . 37
31. Fundamental problems of analytic geometry 40

CHAPTER III

The Locus of an Equation

32. The locus of an equation 43
33. Illustrative examples: Cartesian coördinates 43
34. Loci by polar coördinates 46
35. The locus of an equation 47
36. Classification of loci 48
37. Construction of loci. Discussion of equations 49
38. The locus of an equation remains unchanged: (α) by any transposition of the terms of the equation; and (β) by multiplying both members of the equation by any finite constant . 52
39. Points of intersection of two loci 53
40. Product of two or more equations 54
41. Locus represented by the sum of two equations 56

CHAPTER IV

The Equation of a Locus

42. The equation of a locus 61
43. Equation of straight line through two given points . . . 61

ARTICLE PAGE
44. Equation of straight line through given point and in given direction 63
45. Equation of a circle; polar coördinates 64
46. Equation of locus traced by a moving point 65
47. Equation of a circle: second method 66
48. The conic sections 67
49. The use of curves in applied mathematics 73

CHAPTER V

The Straight Line. Equation of First Degree

$$Ax + By + C = 0$$

50. Recapitulation 81
51. Equation of straight line through two given points . . . 81
52. Equation of straight line in terms of the intercepts which it makes on the coördinate axes 83
53. Equation of straight line through a given point and in a given direction 84
54. Equation of straight line in terms of the perpendicular from the origin upon it, and the angle which that perpendicular makes with the x-axis 86
55. Normal form of equation of straight line: second method . 87
56. Summary 88
57. Every equation of the first degree between two variables has for its locus a straight line 89
58. Reduction of the general equation $Ax + By + C = 0$ to the standard forms. Determination of a, b, m, p, and α in terms of A, B, and C 91
59. To trace the locus of an equation of the first degree . . . 94
60. Special cases of the equation of the straight line $Ax+By+C=0$ 95
61. To find the angle made by one straight line with another . . 97
62. Condition that two lines are parallel or perpendicular . . 98
63. Line which makes a given angle with a given line . . . 101
64. The distance of a given point from a given line . . . 105
65. Bisectors of the angles between two given lines . . . 108
66. The equation of two lines 110
67. Condition that the general quadratic expression may be factored 111
68. Equations of straight lines: coördinate axes oblique . . . 115
69. Equations of straight lines: polar coördinates 118

CHAPTER VI

TRANSFORMATION OF COÖRDINATES

ARTICLE PAGE
70. Introductory 123

I. *Cartesian Coördinates Only*

71. Change of origin, new axes parallel respectively to the original axes. 124
72. Transformation from one system of rectangular axes to another system, also rectangular, and having the same origin; change of direction of axes 126
73. Transformation from rectangular to oblique axes, origin unchanged 127
74. Transformation from one set of oblique axes to another, origin unchanged 128
75. The degree of an equation in Cartesian coördinates is not changed by transformation to other axes 129

II. *Polar Coördinates*

76. Transformations between polar and rectangular systems. . 130

CHAPTER VII

THE CIRCLE

Special Equation of the Second Degree

$$Ax^2 + Ay^2 + 2\,Gx + 2\,Fy + C = 0$$

77. Introductory 135
78. The circle: its definition and equation 135
79. In rectangular coördinates every equation of the form $x^2 + y^2 + 2\,Gx + 2\,Fy + C = 0$ represents a circle 137
80. Equation of a circle through three given points . . . 138

Secants, Tangents, and Normals

81. Definitions of secants, tangents, and normals 140
82. Tangents: Illustrative examples 141
83. Equation of tangent to the circle $x^2 + y^2 = r^2$ in terms of its slope 142
84. Equation of tangent to the circle in terms of the coördinates of the point of contact: the secant method 144

ARTICLE PAGE
85. Equation of a normal to a given circle 147
86. Lengths of tangents and normals. Subtangents and subnormals 149
87. Tangent and normal lengths, subtangent and subnormal, for the circle 150
88. To find the length of a tangent from a given external point to a given circle 151
89. From any point outside of a circle two tangents to the circle can be drawn 152
90. Chord of contact 154
91. Poles and polars 156
92. Equation of the polar 156
93. Fundamental theorem 157
94. Geometrical construction for the polar of a given point, and for the pole of a given line, with regard to a given circle . 158
95. Circles through the intersections of two given circles . . 160
96. Common chord of two circles 160
97. Radical axis; radical center 161
98. The equation of a circle: polar coördinates 162
99. Equation of a circle referred to oblique axes 163
100. The angle formed by two intersecting curves 164

CHAPTER VIII

The Conic Sections

101. Recapitulation 170

I. *The Parabola*

Special Equation of Second Degree

$Ax^2 + 2\,Gx + 2\,Fy + C = 0$, or $By^2 + 2\,Gx + 2\,Fy + C = 0$

102. The parabola defined 170
103. First standard form of the equation of the parabola . . 171
104. To trace the parabola $y^2 = 4\,px$ 172
105. Latus rectum 173
106. Geometric property of the parabola. Second standard equation 173
107. Every equation of the form
$Ax^2 + 2\,Gx + 2\,Fy + C = 0$, or $By^2 + 2\,Gx + 2\,Fy + C = 0$,
represents a parabola whose axis is parallel to one of the coördinate axes 175
108. Reduction of the equation of a parabola to a standard form . 177

II. *The Ellipse*

Special Equation of the Second Degree

$$Ax^2 + By^2 + 2\,Gx + 2\,Fy + C = 0$$

ARTICLE PAGE

109. The ellipse defined 179
110. The first standard equation of the ellipse 180
111. To trace the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ 182
112. Intrinsic property of the ellipse. Second standard equation . 183
113. Every equation of the form $Ax^2 + By^2 + 2\,Gx + 2\,Fy + C = 0$ represents an ellipse whose axes are parallel to the coördinate axes, if A and B have the same sign 186
114. Reduction of the equation of an ellipse to a standard form . 189

III. *The Hyperbola*

Special Equation of the Second Degree

$$Ax^2 - By^2 + 2\,Gx + 2\,Fy + C = 0$$

115. The hyperbola defined 190
116. The first standard form of the equation of the hyperbola . 191
117. To trace the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ 193
118. Intrinsic property of the hyperbola. Second standard equation 195
119. Every equation of the form $Ax^2 + By^2 + 2\,Gx + 2\,Fy + C = 0$ represents an hyperbola whose axes are parallel to the coördinate axes, if A and B have unlike signs 197
120. Summary 199

IV. *Tangents, Normals, Polars, Diameters, etc.*

121. Introductory 200
122. Tangent to the conic $Ax^2 + By^2 + 2\,Gx + 2\,Fy + C = 0$ in terms of the coördinates of the point of contact: the secant method 200
123. Normal to the conic $Ax^2 + By^2 + 2\,Gx + 2\,Fy + C = 0$, at a given point 203
124. Equation of a tangent, and of a normal, that pass through a given point which is not on the conic 205

ARTICLE PAGE

125. Through a given external point two tangents to a conic can be drawn 206
126. Equation of a chord of contact 207
127. Poles and polars 209
128. Fundamental theorem 210
129. Diameter of a conic section 211
130. Equation of a conic that passes through the intersections of two given conics 213

V. *Polar Equation of the Conic Sections*

131. Polar equation of the conic 214
132. From the polar equation of a conic to trace the curve . . 215

CHAPTER IX

THE PARABOLA $y^2 = 4px$

133. Review 219
134. Construction of the parabola 220
135. The equation of the tangent to the parabola $y^2 = 4px$ in terms of its slope 221
136. The equation of the normal to the parabola $y^2 = 4px$ in terms of its slope 222
137. Subtangent and subnormal. Construction of tangent and normal 222
138. Some properties of the parabola which involve tangents and normals 225
139. Diameters 230
140. Some properties of the parabola involving diameters . . 232
141. The equation of a parabola referred to any diameter and the tangent at its extremity as axes 233

CHAPTER X

THE ELLIPSE $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

142. Review 237
143. The equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in terms of its slope 238
144. The sum of the focal distances of any point on an ellipse is constant; it is equal to the major axis 239

ARTICLE PAGE
145. Construction of the ellipse 240
146. Auxiliary circles. Eccentric angle 242
147. The subtangent and subnormal. Construction of tangent and normal 244
148. The tangent and normal bisect externally and internally, respectively, the angles between the focal radii of the point of contact 246
149. The intersection of the tangents at the extremity of a focal chord 247
150. The locus of the foot of the perpendicular from a focus upon a tangent to an ellipse 248
151. The locus of the intersection of two perpendicular tangents to the ellipse 249
152. Diameters 250
153. Conjugate diameters 252
154. Given an extremity of a diameter, to find the extremity of its conjugate diameter 253
155. Properties of conjugate diameters of the ellipse . . . 254
156. Equi-conjugate diameters 257
157. Supplemental chords 259
158. Equation of the ellipse referred to a pair of conjugate diameters 260
159. Ellipse referred to conjugate diameters; second method . 261

CHAPTER XI

THE HYPERBOLA $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

160. Review 265
161. The difference between the focal distances of any point on an hyperbola is constant; it is equal to the transverse axis . 266
162. Construction of the hyperbola 267
163. The tangent and normal bisect internally and externally the angles between the focal radii of the point of contact . . 268
164. Conjugate hyperbolas 270
165. Asymptotes 272
166. Relation between conjugate hyperbolas and their asymptotes 275
167. Equilateral or rectangular hyperbola 277
168. The hyperbola referred to its asymptotes 278
169. The tangent to the hyperbola $xy = c^2$ 280
170. Geometric properties of the hyperbola 281

ARTICLE PAGE
171. Diameters 284
172. Properties of conjugate diameters of the hyperbola . . 285
173. Supplemental chords 287
174. Equations representing an hyperbola, but involving only one variable 288

CHAPTER XII

GENERAL EQUATION OF THE SECOND DEGREE

$$Ax^2 + 2\,Hxy + By^2 + 2\,Gx + 2\,Fy + C = 0$$

175. General equation of the second degree in two variables . . 292
176. Illustrative examples 294
177. Test for the species of a conic 297
178. Center of a conic section 298
179. Transformation of the equation of a conic to parallel axes through its center 299
180. The invariants $A + B$ and $H^2 - AB$ 301
181. To reduce to its simplest standard form the general equation of a conic 303
182. Summary 306
183. The equation of a conic through given points . . . 307

CHAPTER XIII

HIGHER PLANE CURVES

184. Definitions 309

I. *Algebraic Curves*

185. The cissoid of Diocles 309
186. The conchoid of Nicomedes 312
187. The witch of Agnesi 314
188. The lemniscate of Bernouilli 315
189 *a*. The limaçon of Pascal 318
189 *b*. The cardioid 319
190. The Neilian, or semi-cubical parabola 320

II. *Transcendental Curves*

191. The cycloid 321
192. The hypocycloid 323

III. *Spirals*

ARTICLE PAGE
193. Definition 325
194. The spiral of Archimedes 325
195. The reciprocal, or hyperbolic, spiral 326
196. The parabolic spiral 328
197. The lituus or trumpet 328
198. The logarithmic spiral 329

PART II.—SOLID ANALYTIC GEOMETRY

CHAPTER I

Coördinate Systems. The Point

199. Introductory 331
200. Rectangular coördinates 332
201. Polar coördinates 333
202. Relation between the rectangular and polar systems . . . 333
203. Direction angles: direction cosines 334
204. Distance and direction from one point to another; rectangular coördinates 336
205. The point which divides in a given ratio the straight line from one point to another 337
206. Angle between two radii vectores. Angle between two lines 338
207. Transformation of coördinates; rectangular systems . . . 339

CHAPTER II

The Locus of an Equation. Surfaces

208. Introductory 342
209. Equations in one variable. Planes parallel to coördinate planes 343
210. Equations in two variables. Cylinders perpendicular to coördinate planes 344
211. Equations in three variables. Surfaces 346
212. Curves. Traces of surfaces 347
213. Surfaces of revolution 348

CHAPTER III

EQUATIONS OF THE FIRST DEGREE $Ax + By + Cz + D = 0$. PLANES AND STRAIGHT LINES

I. *The Plane*

ARTICLE PAGE
214. Every equation of the first degree represents a plane . . 353
215. Equation of a plane through three given points . . . 354
216. The intercept equation of a plane 354
217. The normal equation of a plane 355
218. Reduction of the general equation of first degree to a standard form. Determination of the constants $a, b, c, p, \alpha, \beta, \gamma$ 356
219. The angle between two planes. Parallel and perpendicular planes 357
220. Distance of a point from a plane 359

II. *The Straight Line*

221. Two equations of the first degree represent a straight line . 359
222. Standard forms for the equations of a straight line
  (*a*) The straight line through a given point in a given direction 360
  (*b*) The straight line through two given points . . . 360
  (*c*) The straight line with given traces on the coördinate planes 361
223. Reduction of the general equations of a straight line to a standard form. Determination of the direction angles and traces
  I. Third standard form: traces 362
  II. First standard form: direction angles 362
224. The angle between two lines; between a plane and a line . 363

CHAPTER IV

EQUATIONS OF THE SECOND DEGREE. QUADRIC SURFACES

225. The locus of an equation of second degree 367
226. Species of quadrics. Simplified equation of second degree . 368
227. Standard forms of the equation of a quadric 370
228. The ellipsoid: equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ 371

ARTICLE PAGE

229. The un-parted hyperboloid: equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ . . 373

230. The bi-parted hyperboloid: equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ . . 375

231. The paraboloids: equation $\frac{x^2}{a^2}\pm\frac{y^2}{b^2}=z$ 376

232. The cone: equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$ 378

233. The hyperboloid and its asymptotic cone 379

APPENDIX

NOTE A. Historical sketch 381

NOTE B. Construction of any conic 382

NOTE C. Special cases of the conics 383

NOTE D. Every section of a cone by a plane is a conic . . . 384

NOTE E. Parabola as a limiting form of ellipse or hyperbola . . 387

NOTE F. Confocal conics 388

ANSWERS 391

INDEX 000

ANALYTIC GEOMETRY

◆

PART I

◆

CHAPTER I

INTRODUCTION

ALGEBRAIC AND TRIGONOMETRIC CONCEPTIONS

1. Number. A number is most simply interpreted as expressing the measurement of one quantity by another quantity of the same kind first chosen as a unit of measure; it is **positive,** or +, if the measuring unit is taken in the same sense as the thing measured; and **negative,** or −, if this measuring unit is taken in the opposite sense.

E.g., the unit *dollar* may be regarded as a dollar of assets, or as a dollar of liabilities; if it is regarded as a dollar of assets, then assets measured by it produce positive numbers, while liabilities measured by it produce negative numbers.

The above definition is consistent with the one usually given; viz. that numbers are positive or negative according as they are greater or less than zero.

If the operations of addition, subtraction, multiplication, division, raising to integer powers, extracting roots, or any combination of these operations, are performed upon given numbers, the result in every case is a number; it is **imaginary** if it involves in any way whatever an indicated even root of a negative number; otherwise it is **real.**

Every imaginary number may be reduced to the form $a + b\sqrt{-1}$, where a and b are real, and $b \neq 0$.

2. Constants and variables. If AB and AC are two given straight lines making an angle α at the point A, and if any two points X and Y, on these lines, respectively, are joined by a straight line, then

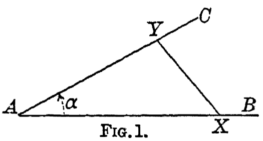


Fig. 1.

$$\text{Area of triangle } AXY = \tfrac{1}{2} \cdot AX \cdot AY \cdot \sin\alpha,$$

i.e.,
$$\Delta = \tfrac{1}{2} \cdot x \cdot y \cdot \sin\alpha,$$

where x is the length of AX, y is the length of AY, and Δ is the area of the triangle.

If now the points X and Y are moved along the lines AB and AC in any way whatever, then Δ, x, and y will each pass through a series of different values, — they are **variable numbers** or **variables**; while $\frac{1}{2}$ and $\sin\alpha$ will remain unchanged, — they are **constant numbers** or **constants.**

It is to be remarked that $\frac{1}{2}$ has the same value wherever it occurs, — it is an *absolute* constant; while α, though constant for this series of triangles, may have a different constant value for another series of triangles, — it is an *arbitrary* constant.

Because x and y may separately take any values whatever they are *independent* variables; while Δ, whose value depends upon the values of x and y, is a *dependent* variable.

The illustrations just given may serve to give a clearer conception of the following more formal definitions.

An **absolute constant** is a number which has the same value wherever it occurs; such are the numbers 2, 7, $\frac{3}{5}$, $6^{\frac{2}{3}}$, π, e

(where $\pi = 3.14159265\cdots$, approximately $\frac{22}{7}$, the ratio of the circumference of a circle to its diameter; and

$$e = 2.71828182\cdots = 1 + \frac{1}{1} + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \cdots,$$

approximately $\frac{19}{7}$, the base of the Naperian system of logarithms).

An **arbitrary constant** is a number which retains the same value throughout the investigation of a given problem, but may have a different fixed value in another problem.

An **independent variable** is a number that may take any value whatever within limits prescribed by the conditions of the problem under consideration.

A **dependent variable** is a number that depends for its value upon the values assumed by one or more independent variables.*

A number that is greater than any assignable number, however great, is an **infinite number**; one that varies and becomes and remains smaller (numerically, not merely algebraically *less*) than any assigned number, however small, is an **infinitesimal number.** All other numbers are **finite.**

3. Functions. A number so related to one or more other numbers that it depends upon these for its value, and takes in general a definite value, or a finite number of definite values, when each of these other numbers takes a definite value, is a **function** of these other numbers. *E.g.*, the circumference and the area of a circle are functions of its radius; the distance traveled by a railway train is a function of its time and rate; if $y = 3x^2 + 5x - 8$, then y is a function of x.

---

* All these kinds of numbers will be met and better illustrated in succeeding chapters of this book. *E.g.*, see Art. 55, Note.

4. Identity, equation, and root. If two functions involving the same variables are equal to each other for *all* values of those variables they are *identically* equal. Such an equality is expressed by writing the sign $\equiv$ between the two functions, and the expression so formed is an **identity**. If, on the other hand, the two functions are equal to each other only for particular values of the variables, the equality is expressed by writing the sign $=$ between the two functions, and the expression so formed is an **equation**. The particular values for which the two functions are equal, *i.e.*, those values of the variables which *satisfy* the equation, are the **roots** of the equation.

E.g., $(x+y)^2 \equiv x^2+2xy+y^2$, $(x+a)(x-a)+a^2 \equiv x^2$,

and
$$x+\frac{3}{x-1} \equiv \frac{x^2-x+3}{x-1}$$

are identities; while $3x^2-10x+2=2x^2-4x-6$, or, what is the same thing, $x^2-6x+8=0$, is an equation. The roots of this equation are the numbers 2 and 4.

Special attention is called to the fact that an equation always imposes a *condition*.

E.g., $x^2-6x+8=0$ if, and only if, $x=2$ or $x=4$. So also the equation $ax+by+c=0$ imposes the condition that x shall be equal to
$$\frac{-by-c}{a}.$$

5. Functions classified. A functional relation is usually expressed by means of an equation involving the related numbers. If the form of this equation is such that one of the variables is expressed directly in terms of the others, then that variable is called an **explicit** function of the others; if it is not so expressed, it is an **implicit** function.

E.g., the equations $y=\sqrt{5-x^2}$, $x^2+y^2=5$, and $x=\sqrt{5-y^2}$ express the same relation between x and y; in the first y is an explicit function of x, in the second each is an implicit function of the other, while in the third x is an explicit function of y.

The word "function" is, for brevity, usually represented by a single letter, such as f, F, ϕ, $\psi, \cdots$; thus $y = \phi(x)$ means that y is a function of the independent variable x, and is read "y equals the ϕ-function of x"; so also $z = F(u, v, x)$ means that z is a function of the independent variables u, v, and x, and it is read, "z equals the F-function of u, v, and x."

A function is **algebraic** if it involves, so far as the independent variables are concerned, only a finite number of the operations of addition, subtraction, multiplication, division, raising to integer powers, and extracting roots. All other functions are **transcendental.**

E.g., $2x^3 - 5x - 17$, $xy + y^2 - 7x$, and $\dfrac{2x^2 - 11y^2}{x + xy - 7y^2}$ are algebraic functions; while 2^y, a^x, $\sin x$, $\tan^{-1} z$, and $\log t$ are transcendental functions.

6. Notation. In general, absolute constants are represented by the Arabic numerals, while arbitrary constants and variables are represented by letters. A few absolute constants are, however, by general consent, represented by letters; examples of such constants are π and e (Art. 2). Variables are usually represented by the last letters of the alphabet, such as u, v, w, x, y, z; while the first letters, a, b, c, $\cdots$ are reserved to represent constants.

Particular fixed values from among those that a variable may assume are sometimes in question; *e.g.*, the values, $x = 2$ and $x = -1$, for which the function $x^2 - x - 2$ vanishes; such values may conveniently be denoted by affixing a subscript to the letter representing the variable. Thus $x_1, x_2, x_3, \cdots$ will be used to denote particular values of the variable x.

Similarly, variables which enter a problem in analogous ways are usually denoted by a single letter having accents attached to it; thus x', x'', x''', $\cdots$ denote variables that are similarly involved in a given problem.

Again, each of the two equations, $y=3x^2-4x+10$ and $y=\phi(x)$, asserts that y is a function of x; but while the former tells precisely *how* y depends upon x, the latter merely asserts that *there is* such a dependence, without giving any information concerning the form of that dependence. If several different forms of functions present themselves in the same problem, they are represented by different letters, each letter representing a particular form for that problem, though it may be chosen to represent an entirely different form in another problem.

E.g., if the form of ϕ, in a given problem, is defined by the equation

$$\phi(x)=\frac{3x^5-x^4+5}{2x+1},$$

then, *in the same problem*,

$$\phi(v)=\frac{3v^5-v^4+5}{2v+1},\quad \phi(1)=\frac{7}{3},\quad \text{and}\quad \phi(0)=5.$$

7. Continuous and discontinuous functions. In general a function takes different values when different values are assigned to its independent variable. If $y=\phi(x)$, then, for $x=a$ and $x=b$, the function becomes $y_1=\phi(a)$ and $y_2=\phi(b)$, and y_1 is in general different from y_2. The function $\phi(x)$ is said to be a **continuous** function of x between $x=a$ and $x=b$, if, while x is made to pass successively through all real values from a to b, y remains real and finite and passes correspondingly through all values from y_1 to y_2.

This definition may be more precisely stated, thus: If x_1 and x_2 are any real values of x which lie between the values a and b, and if the corresponding values of y, viz. $\phi(x_1)$ and $\phi(x_2)$, are real and finite; and if a positive number η can be found, such that by taking, numerically,

$$x_1 - x_2 < \eta,$$

it will follow that, numerically,

$$\phi(x_1) - \phi(x_2) < \epsilon,$$

where ϵ is any assigned positive number, however small; then $\phi(x)$ is a continuous function of x for values from a to b.

Or, in words: y is a continuous function of x for all values of x in the interval from a to b, if, by taking any two values of x in the interval sufficiently near together, the difference between the corresponding values of y can be made less than any assigned number, however small.

A **discontinuous function** is one that does not fulfil the conditions for continuity. It is, however, *usually* discontinuous for only a limited number of particular values of its independent variable, while between these values it is continuous.

As familiar examples of continuous functions may be mentioned: the length of a solar shadow; the area of a cross-section of a growing tree, or of a growing peach; the height of the mercury in a barometer; the temperature of a room at varying distances from the source of heat; and interest as a function of time.

So, also, $y = 3x^2 + 4x + 1$ is a continuous function of x for all finite values of x.

For, y remains real and finite so long as x remains real and finite, and, if x_1 and x_2 be any two finite values of x which differ from each other by η, *i.e.*, if $x_2 = x_1 \pm \eta$, then

$$\begin{aligned} y_2 - y_1 &= 3x_2^2 + 4x_2 + 1 - (3x_1^2 + 4x_1 + 1), \\ &= 3(x_1 \pm \eta)^2 + 4(x_1 \pm \eta) + 1 - (3x_1^2 + 4x_1 + 1), \\ &= \pm (6x_1 + 4 + 3\eta)\eta. \end{aligned}$$

Now to show that $y = 3x^2 + 4x + 1$ is continuous for $x = x_1$, it only remains to show that, by taking η sufficiently small, *i.e.*, by taking x_2 sufficiently near x_1, y_2 can be made to differ from y_1 by less than any assigned number (ϵ), however small. But this is evident; for η may be taken as near zero as desired, hence the factor $6x_1 + 4 + 3\eta$ as near $6x_1 + 4$ as desired, and the product therefore as near zero as is necessary to be less than ϵ.

On the other hand, if, at regular intervals of time, apples are dropped into a basket, the combined weight of the basket and apples will increase discontinuously; *i.e.*, their total weight is a discontinuous function of the time.

EXERCISES

1. If $Ax + By + C = 0$, prove that y is a continuous function of x; and x, of y.

2. If $x^2 + y^2 - 4 = 0$, prove that y is a continuous function of x, when $2 > x > -2$.

3. If $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, prove that x is a continuous function of y, when $b > y > -b$.

4. If $\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$, is x a continuous function of y?

5. If $st - 9 = 0$, is s a continuous function of t?

6. If $u^2 - 3v = 0$, is u a continuous function of v? Is v a continuous function of u?

7. Show that all functions of the form

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n,$$

where $a_0, a_1, a_2 \cdots a_n$ are constants, are continuous for all finite values of x.

8. If $\frac{y-1}{y-2} = 5^{\frac{1}{x-1}}$, show that y is discontinuous for $x = 1$.

9. Find the value of x for which $y, = c\,\dfrac{e^{\frac{1}{x-a}} - 1}{e^{\frac{1}{x-a}} + 1}$, is discontinuous.

10. Interest on money loaned is calculated by the formula

$$I = P \cdot R \cdot T.$$

Is the interest (I) a continuous or a discontinuous function of P? of R? of T?

8. The present work will be concerned for the most part with algebraic functions involving only the first and second powers of the variable, *i.e.*, with algebraic equations of the first and second degree. A review is therefore given of the solution and theory of the quadratic equation, presenting in brief the most important results which will be needed in the Analytic Geometry. The student should become thoroughly familiar with this theory, as well as with the review of the trigonometry which follows it.

9. The quadratic equation. Its solution. The most general equation of the second degree, in one unknown number, may be written in the form

$$ax^2 + bx + c = 0, \quad . \quad . \quad . \quad (1)$$

where a, b, and c are known numbers. This equation may be solved by the method of "completing the square," which gives

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}, \quad . \quad . \quad . \quad (2)$$

i.e., $$x + \frac{b}{2a} = \pm\sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}} = \pm\frac{1}{2a}\sqrt{b^2 - 4ac}, \quad . \quad . \quad . \quad (3)$$

whence $$x = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}. \quad . \quad . \quad . \quad (4)$$

If x_1 and x_2 are used to denote the roots of eq. (1), they may be written

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \text{ and } x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \quad . \quad . \quad . \quad (5)$$

The nature of the roots (5) depends upon the number under the radical sign, *i.e.*, upon $b^2 - 4ac$, giving three cases to be considered, viz.:

$$\left.\begin{array}{l}\text{if } b^2-4ac>0, \text{ then the roots are both real and unequal,}\\ \text{if } b^2-4ac=0, \text{ then the roots are both real and equal,}\\ \text{if } b^2-4ac<0, \text{ then the roots are both imaginary.}\end{array}\right\} \quad (6)$$

Thus the *character* of the roots of a given quadratic equation may be determined without actually solving the equation, by merely calculating the value of the expression $b^2 - 4ac$. This important expression is called the **discriminant** of the quadratic equation; when equated to zero it states the *condition* that must hold among the coefficients if the equation has equal roots.

EXERCISES

1. Show which of the following equalities are identities:

(1) $x^2 - 4x + 4 = 0$;

(2) $(s + t)(s - t) = s^2 - t^2$;

(3) $\dfrac{\alpha^3 + \beta^3}{\alpha + \beta} = \alpha^2 - \alpha\beta + \beta^2$;

(4) $(p + q)^3 = p^3 + q^3 + 3pq(p + q)$;

(5) $x^2 + 5x + 6 = (x + 3)(x + 2)$.

2. Determine, without solving the equation, the nature of the roots of

$$3x^2 + 8x + 1 = 0.$$

Solution. Since $b^2 - 4ac = 64 - 12 = 52$, *i.e.*, is positive, therefore the roots are real and unequal; again, since a, b, and c are all positive, therefore both roots are negative (cf. eq. (4), Art. 9).

3. Without solving the equation, determine the character of the roots of $8x^2 - 3x + 1 = 0$.

4. Given the equation $x^2 - 3x - m(x + 2x^2 + 4) = 5x^2 + 3$. Find the roots. For what values of m are these roots equal?

5. Determine, without solving, the character of the roots of the equations:

(1) $5z^2 - 2z + 5 = 0$; (2) $x^2 + 7 = 0$; (3) $3t^2 - t = 19$.

6. Determine the values of m for which the following equations shall have equal roots:

(1) $x^2 - 2x(1+3m) + 7(3+2m) = 0$;
(2) $mx^2 + 2x^2 - 2m = 3mx - 9x + 10$;
(3) $4x^2 + (1+m)x + 1 = 0$; (4) $x^2 + (6x+m)^2 = a^2$.

7. If in the equation $2ax(ax+nc) + (n^2-2)c^2 = 0$, x is real, show that n lies between -2 and $+2$.

8. If x is real in the equation $\frac{x}{x^2-5x+9} = a$, show that a lies between 1 and $-\frac{1}{11}$.

9. For what values of c will the following equations have equal roots?

(1) $3x^2+4x+c=0$; (2) $(mx+c)^2=4lx$; (3) $4x^2+9(2x+c)^2=36$.

10. Solve the equations in examples 2, 3, and 5.

11. Solve the equations:

(1) $x^4 - 25x^2 = -144$; (2) $\frac{3x-2}{x-2} - \frac{2x+1}{x+2} + \frac{12}{x^2-4} = 0$.

10. Zero and infinite roots. In the following pages it will sometimes be necessary to know the conditions among the coefficients of a quadratic equation that will make one or both of its roots zero, or the conditions that will make one or both of the roots infinitely large. In equations (5) of Art. 9, x_1 and x_2, *i.e.* the roots of $ax^2 + bx + c = 0$, were found; and it is at once seen that

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-b+\sqrt{b^2-4ac}}{2a} \cdot \frac{-b-\sqrt{b^2-4ac}}{-b-\sqrt{b^2-4ac}} = \frac{2c}{-b-\sqrt{b^2-4ac}}, \qquad (1)$$

and that

$$x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a} = \frac{2c}{-b+\sqrt{b^2-4ac}}. \quad . \quad . \quad . \qquad (2)$$

Equations (1) and (2) show that:

(1) If a and b remain unchanged while c grows smaller,

then x_1 grows smaller and x_2 grows larger; and if $c \doteq 0$,* then $x_1 \doteq 0$, while $x_2 \doteq -\frac{b}{a}$.

(2) If a remains unchanged while $c \doteq 0$ and $b \doteq 0$, then $x_1 \doteq 0$ and $x_2 \doteq 0$.

(3) If b and c remain unchanged while $a \doteq 0$, then $x_1 \doteq -\frac{c}{b}$ and x_2 becomes infinitely large.

(4) If c remains unchanged while $a \doteq 0$ and $b \doteq 0$, then both x_1 and x_2 become infinitely large.

(5) If a and c remain unchanged while $b \doteq 0$, then $x_1 \doteq \sqrt{\frac{-c}{a}}$ and $x_2 \doteq -\sqrt{\frac{-c}{a}}$.

The student should translate (1), (2), (3), (4), and (5) into more general terms by reading "the absolute term approaches zero as a limit" instead of "$c \doteq 0$," etc.

11. Properties of the quadratic equation. By adding the two roots of

$$ax^2 + bx + c = 0 \quad . \quad . \quad . \quad (1)$$

and also multiplying them together, the relations

$$x_1 + x_2 = -\frac{b}{a} \text{ and } x_1x_2 = \frac{c}{a} \quad . \quad . \quad . \quad (2)$$

are obtained; or, if equation (1) is written with the coefficient of the term of the second degree reduced to unity, as

$$x^2 + px + q = 0, \quad . \quad . \quad . \quad (3)$$

these relations become

$$x_1 + x_2 = -p \text{ and } x_1x_2 = q, \quad . \quad . \quad . \quad (4)$$

Or, expressed in words: the coefficient of the term of the second degree being unity, the coefficient of the term of the first degree is the negative of the sum of the roots, while the term free from x is the product of the roots.

---

* The sign $\doteq$ is read "approaches as a limit." It was introduced by the late Professor Oliver of Cornell University.

If, therefore, the roots of a quadratic equation are not themselves needed, but only their sum or product is desired, these may be obtained directly from the given equation by inspection.

E.g., the half sum of the roots of the equation

$$m^2x^2 + 2(bm - 2\,l)x + b^2 = 0$$

is
$$\frac{x_1 + x_2}{2} = -\frac{2(bm - 2\,l)}{2\,m^2} = \frac{2\,l - bm}{m^2}.$$

Moreover, if x_1 and x_2 are the roots of the equation

$$x^2 + px + q = 0,$$

then $x - x_1$ and $x - x_2$ are the factors of its first member.

For, by equation (4) above, this equation may be written

$$x^2 + px + q \equiv x^2 - (x_1 + x_2)x + x_1x_2 = 0,$$

and
$$x^2 - (x_1 + x_2)x + x_1x_2 \equiv (x - x_1)(x - x_2),$$

hence
$$x^2 + px + q \equiv (x - x_1)(x - x_2).$$

Conversely: if a quadratic function can be separated into two factors of the first degree, then the roots can be immediately written by inspection.

For, if $x^2 + px + q \equiv (x - x_1)(x - x_2)$, then the first member will vanish if, and only if, $x - x_1 = 0$ or $x - x_2 = 0$; *i.e.* $x^2 + px + q = 0$ if $x = x_1$ or $x = x_2$, hence x_1 and x_2 are the roots of the equation $x^2 + px + q = 0$ (cf. Art. 4).

12. The quadratic equation involving two unknowns. One equation involving two unknown numbers cannot be solved uniquely for the values of those numbers which satisfy the equation; but if there is assigned to either of those numbers a definite value, then at least one definite and corresponding value can be found for the other, so that, this *pair* of values being substituted for the unknown numbers, the equation will be satisfied. In this way an infinite number of pairs of values, that will satisfy the equation, may be found.

If, however, the equation is *homogeneous* in the two unknowns, *i.e.*, of the form

$$ax^2 + bxy + cy^2 = 0,$$

then the ratio $x : y$ may be regarded as a single number, and the equation has properties precisely like those discussed in Arts. 9, 10, and 11.

To solve a system consisting of two or more independent simultaneous equations, involving as many unknown elements, it is necessary to combine the equations so as to eliminate all but one of the unknown elements, then to solve the resulting equation for that one, and, by means of the roots thus obtained, find the entire system of roots.

EXERCISES

1. Given the equation $x^2 + 3x - 4 + m(3x^2 - 4) - 2mx^2 = 0$, find the sum of the roots; the product of the roots; also the factors of the first member.

2. Factor the following expressions:

(1) $x^2 - 5x + 4$; (3) $mx^2 - 3x + c$; (5) $3w^{\frac{5}{3}} - 94w^{\frac{5}{6}} - 64$;
(2) $x^2 + 2x - 8$; (4) $ax^2 + bxy + cy^2$; (6) $11 - 27y - 18y^2$.

3. Without first solving the equation

$$x^2 - 3x - m(x + 2x^2 + 4) = 5x^2 + 3$$

find the sum, and the product, of its roots. For what value of m are its roots equal? For what value of m do both its roots become infinitely large? If all the terms are transposed to one member, what are the factors of that member?

4. Without first solving, determine the nature of the roots of the equation $(m - 2)(\log x)^2 - (2m + 3)\log x - 4m = 0$. [Regard $\log x$ as the unknown element.]

For what values of m are the roots equal? Real? One infinitely great? Both infinitely great? One zero? Find the factors of the first member of the equation.

5. Find five pairs of numbers that satisfy the equation:

(1) $x + 3y - 7 = 0$; (3) $y^2 = 16x$;
(2) $x^2 + y^2 = 4$; (4) $3x + 6xy - 8y^2 + 3x^2 = 0$.

6. Without solving, determine the nature of the roots of the equation:

$$9x^2 + 12xy + 4y^2 = 0, \quad 3u^2 - uv + 19v^2 = 0.$$

7. Solve the following pairs of simultaneous equations:

(1) $3x - 5y + 2 = 0$, and $2x + 7y - 4 = 0$;
(2) $5y + 2z + 3 = 0$, and $7y + 4z + 2 = 0$;
(3) $y = 3x + c = 0$, and $y^2 = 9x$;
(4) $x^2 + y^2 = 5$, and $y^2 = 6x$;
(5) $b^2x^2 + a^2y^2 = a^2b^2$, and $y = ax + b$;
(6) $\frac{x^2}{16} + \frac{y^2}{9} = 1$, and $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

8. Determine those values of b for which each of the following pairs of equations will be satisfied by two equal values of y:

(1) $\{x^2 + y^2 = a^2,\ y = 6x + b\}$; (2) $\{y = mx + b,\ y^2 = 4x\}$;
(3) $\{3y + 2x = b,\ 6x^2 + y^2 = 12\}$.

9. Determine, for the pairs of equations in Ex. 8, those values of b which will give equal values of x.

TRIGONOMETRIC CONCEPTIONS AND FORMULAS

13. Directed lines. Angles. A line is said to be **directed** when a distinction is made between the segment from any point A of the line to another point B, and the opposite segment from B to A. One of these directions is chosen as positive, or +, and the opposite direction is then negative or −.

The **angle** formed by two intersecting directed straight lines is that relation between the positions of the two lines which is expressed by the amount of rotation about their point of intersection necessary to bring the positive end of the initial side into coincidence with the positive end of the terminal side. The point in which the lines intersect is called the **vertex** of the angle. The angle is *positive*, or +, if the rotation from the initial to the terminal side is in *counter-clockwise* direction; the angle is *negative*, or −, if the rotation is *clockwise.*

The angle formed by two directed straight lines in space, which do not meet, is equal to the angle between two intersecting lines, which are respectively parallel to the given lines.

For the measurement of angles there are two absolute units:

(1) *The angular magnitude about a point in a plane*, i.e., a complete revolution. One fourth of a complete revolution is called a **right angle,** $\frac{1}{90}$ of a right angle is a **degree** (1°), $\frac{1}{60}$ of a degree is a **minute** $(1')$, and $\frac{1}{60}$ of a minute is a **second** $(1'')$;

(2) *the angle whose subtending circular arc is equal in length to the radius of that arc;* this angle is called a **radian** $\}1^{(\ )}\{$; it is independent of the length of the radius. Since $\frac{\text{circumference}}{\text{diameter}}=\frac{\text{semi-circumference}}{\text{radius}}=\pi$, it follows that the angle formed by a half rotation, *i.e.*, 180°, is π radians;

i.e., $$180^\circ=\pi^{(r)}=\left(\frac{22}{7}\right)^{(r)} \text{ approximately};$$

also $$1^{(r)}=\frac{180^\circ}{\pi}=57^\circ\,17'\,44.8'' \text{ approximately.}$$

A right angle is 90° or $\left(\frac{\pi}{2}\right)^{(r)}$.

When there is no danger of being misunderstood, the index (r) is omitted, and $\frac{\pi}{2}$ radians is written simply as $\frac{\pi}{2}$, and not $\left(\frac{\pi}{2}\right)^{(r)}$.

14. Trigonometric ratios. If from any point P in the terminal side of an angle θ, at a distance r from the vertex, a perpendicular MP is drawn to the initial side meeting it in

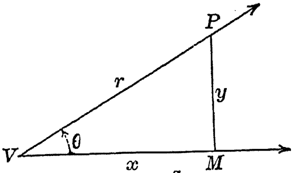


FIG. 2.a

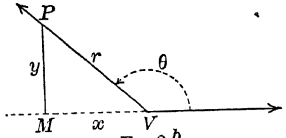


FIG. 2.b

M, and if MP be represented by y and VM by x, then, by general agreement, y is + if MP makes a positive right angle with the initial line, and − if this right angle is negative; similarly, x is + if VM extends in the positive direction of the initial line, and − if it extends in the opposite direction.

The three numbers r, x, and y form with each other six ratios; these ratios, moreover, depend for their value solely upon the size of the angle θ, and not at all upon the value of r. These six ratios are known as the **trigonometric ratios** or **functions** of the angle θ, and are named as follows:

$$\text{sine } \theta = \frac{y}{r}, \qquad \text{tangent } \theta = \frac{y}{x}, \qquad \text{secant } \theta = \frac{r}{x},$$

$$\text{cosine } \theta = \frac{x}{r}, \qquad \text{cotangent } \theta = \frac{x}{y}, \qquad \text{cosecant } \theta = \frac{r}{y}.$$

The abbreviated symbols for these functions are $\sin\theta$, $\cos\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$, and $\csc\theta$, respectively. The functions are not all independent, but are connected by the following relations:

(1) $\sin\theta \cdot \csc = 1$,
(2) $\cos\theta \cdot \sec\theta = 1$,
(3) $\tan\theta \cdot \cot\theta = 1$,
(4) $\tan\theta = \sin\theta : \cos\theta$,
(5) $\cot\theta = \cos\theta : \sin\theta$,
(6) $\sin^2\theta + \cos^2\theta = 1$,
(7) $\tan^2\theta + 1 = \sec^2\theta$,
(8) $\cot^2\theta + 1 = \csc^2\theta$.

By means of these eight relations all the trigonometric functions of any angle may be expressed in terms of any given function. *E.g.*, suppose the sine of an angle is given, and the tangent of this angle, in terms of the sine, is wanted:

by (4), $$\tan\theta = \frac{\sin\theta}{\cos\theta},$$

and by (6), $$\cos\theta = \sqrt{1-\sin^2\theta},$$

hence $$\tan\theta = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}}.$$

If the numerical value of $\sin\theta$ is given, this last formula gives the corresponding numerical value of $\tan\theta$; *e.g.*, if $\sin\theta = \frac{3}{5}$, then

$$\tan\theta = \frac{\frac{3}{5}}{\sqrt{1-(\frac{3}{5})^2}} = \pm\frac{3}{4}.$$

15. Functions of related angles. Based upon the definitions of the trigonometric functions the following relations are readily established.

If θ is any plane angle, then*

(1) $$\sin(-\theta) = -\sin\theta, \quad \cos(-\theta) = +\cos\theta,$$
$$\tan(-\theta) = -\tan\theta, \quad \csc(-\theta) = -\csc\theta,$$
$$\sec(-\theta) = +\sec\theta, \quad \cot(-\theta) = -\cot\theta;$$

(2) $$\sin(\pi\pm\theta) = \mp\sin\theta, \quad \cos(\pi\pm\theta) = -\cos\theta,$$
$$\tan(\pi\pm\theta) = \pm\tan\theta, \quad \csc(\pi\pm\theta) = \mp\csc\theta,$$
$$\sec(\pi\pm\theta) = -\sec\theta, \quad \cot(\pi\pm\theta) = \pm\cot\theta;$$

(3) $$\sin\left(\frac{\pi}{2}\pm\theta\right) = +\cos\theta, \quad \cos\left(\frac{\pi}{2}\pm\theta\right) = \mp\sin\theta,$$
$$\tan\left(\frac{\pi}{2}\mp\theta\right) = \mp\cot\theta, \quad \csc\left(\frac{\pi}{2}\pm\theta\right) = +\sec\theta,$$
$$\sec\left(\frac{\pi}{2}\mp\theta\right) = \mp\csc\theta, \quad \cot\left(\frac{\pi}{2}\pm\theta\right) = \mp\tan\theta.$$

---

* The student should thoroughly familiarize himself with these formulas, and those of Art. 16, as well as with the derivation of each.

16. Other important formulas. If θ_1 and θ_2 are any two plane angles, then

$$\sin(\theta_1 \pm \theta_2) = \sin\theta_1 \cos\theta_2 \pm \cos\theta_1 \sin\theta_2,$$
$$\cos(\theta_1 \pm \theta_2) = \cos\theta_1 \cos\theta_2 \mp \sin\theta_1 \sin\theta_2,$$
$$\tan(\theta_1 \pm \theta_2) = \frac{\tan\theta_1 \pm \tan\theta_2}{1 \mp \tan\theta_1 \tan\theta_2}.$$

If θ is any plane angle, then

$$\sin 2\theta = 2\sin\theta\cos\theta,$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1,$$
$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta},$$
$$\sin\frac{\theta}{2} = \sqrt{\tfrac{1}{2}(1-\cos\theta)},$$
$$\cos\frac{\theta}{2} = \sqrt{\tfrac{1}{2}(1+\cos\theta)},$$
$$\tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta}.$$

If a, b, and c are the sides of a triangle lying respectively opposite the angles A, B, and C, and if Δ is the area of this triangle, then

$$a^2 = b^2 + c^2 - 2bc\cos A, \quad \text{and} \quad \Delta = \tfrac{1}{2}bc\sin A.$$

EXERCISES

1. Express in radians the angles:
$15°$; $60°$; $135°$; $-252°$; $\frac{5}{4}$ rt. angle; $10°\,10'\,10''$; $88°\,2'$; $(3\pi)°$.

2. Express in degrees, minutes, and seconds, the angles:
$\left(\frac{\pi}{4}\right)^{(r)}$; $\left(\frac{3\pi}{5}\right)^{(r)}$; $\left(\frac{1}{4}\right)^{(r)}$; $\left(\frac{2}{9}\right)^{(r)}$; $\frac{7}{10}$ of a revolution; $\frac{5}{4}$ rt. angle.

3. Find the values of the other trigonometric functions, given:
(1) $\tan\theta = 3$; (2) $\sec x = -\sqrt{2}$; (3) $\cos\phi = \frac{1}{\sqrt{3}}$; (4) $\sin t = \frac{1}{7}$; (5) $\cot\psi = \frac{1}{7}$; and (6) $\csc u = -2$.

Solution of (1). If $\tan\theta = 3$, then substituting this value in (3) of Art. 14, gives $\cot\theta = \frac{1}{3}$; substituting these values in (7) and (8) of the same article gives the values of $\sec\theta$ and of $\csc\theta$; and substituting those values in (1) and (2) gives $\sin\theta$ and $\cos\theta$.

Another method: Construct a right triangle ABC with the sides $AB = 1$ and $BC = 3$, then $\angle BAC$ is an angle whose tangent is 3. If $AB = 1$ and $BC = 3$, then $AC = \sqrt{10}$, and the other functions of the angle BAC are at once seen to be:

$$\sin\theta = \frac{3}{\sqrt{10}}, \quad \cos\theta = \frac{1}{\sqrt{10}}, \quad \csc\theta = \frac{\sqrt{10}}{3},$$

$$\sec\theta = \sqrt{10}, \text{ and } \cot\theta = \tfrac{1}{3}.$$

Either of these methods may be employed to solve the other parts of this example; the second method is usually to be preferred.

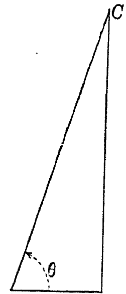


FIG. 3.

4. By means of a right triangle, with appropriate acute angles, find the numerical values of the trigonometric ratios of the following angles:

30°; 45°; 60°; 90°; 135°; and − 45°.

5. Express the following functions in terms of functions of positive angles less than 90° :

$$\tan 3500°; \quad -\csc 290°; \quad \sin(-369°); \quad -\cos\frac{11\pi}{5}; \text{ and } \cot(-1215°).$$

6. Solve the following equations:

$$(1)\ \sin\theta = -\cos 210°; \quad (2)\ \cos\theta = \sin 2\theta; \quad (3)\ \frac{\cos x}{\sin x \cot^2 x} = \sqrt{3};$$

$$\text{and } (4)\ (\sec^2 x - 1)(\csc^2 x + 1) = \tfrac{5}{3}.$$

7. In the following identities transform the first member into the second:

$$(1)\ \frac{\tan\theta - \cot\theta}{\tan\theta + \cot\theta} \equiv \frac{2}{\csc^2\theta} - 1; \qquad (2)\ \frac{\sec x + \csc x}{\sec x - \csc x} \equiv \frac{1 + \cot x}{1 - \cot x};$$

$$(3)\ \csc x(\sec x - 1) - \cot x(1 - \cos x) \equiv \tan x - \sin x;$$

$$(4)\ (2r\sin a\cos a)^2 + r^2(\cos^2 a - \sin^2 a)^2 \equiv r^2;$$

$$(5)\ (\cos a\cos b + \sin a\sin b)^2 + (\sin a\cos b - \cos a\sin b)^2 \equiv 1; \text{ and}$$

$$(6)\ (r\cos\phi)^2 + (r\sin\phi\cos\theta)^2 + (r\sin\phi\sin\theta)^2 \equiv 1.$$

17. Orthogonal projection. The orthogonal projection* of a point upon a line is the foot of the perpendicular from the point to the line. In the figure, M is the projection of P upon AB. The projection of a segment PQ of a line upon another line AB, is that part of the second line extending from the projection of the initial point of the segment to the projection of the terminal point of the segment. Thus MN is the projection of PQ upon AB, and NM is the projection of QP upon AB.

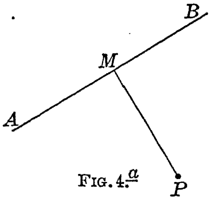


FIG. 4.a

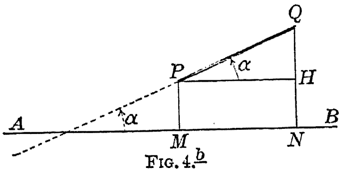


FIG. 4.b

The length of the projection can easily be expressed in terms of the length of the segment and the angle which it makes with the line upon which the segment is projected; for

$$\frac{MN}{PQ}=\frac{PH}{PQ}=\cos\alpha,$$

$$\therefore\ MN=PQ\cdot\cos\alpha;$$

i.e., *the projection of a segment of a line upon another line is equal to the product of its length by the cosine of the angle which it makes with that other line.*

A line made up of parts PQ, QR, RS, ⋯ (Fig. 5a, 5b), which are straight lines having different directions, is a **broken line**; and the projection of a broken line upon any line is the algebraic sum of the projections of its parts upon the same line. Thus the projection of $PQRST$ upon AB is the projection of PQ + the projection of $QR + \cdots$, upon AB; *i.e.*,

---

* Hereafter, unless otherwise stated, *projection* will be understood to mean *orthogonal projection.*

$$\text{proj. } PQRST \text{ upon } AB = MN + NK + KL + LH = MH;$$

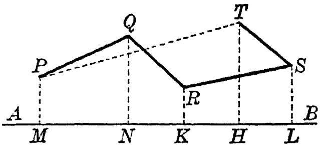


FIG. 5.a

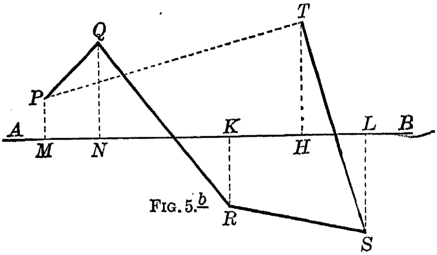


FIG. 5.b

but MH is the projection of the straight line PT which joins the first initial to last terminal point of the broken line. In the same way it may be shown that the projection of any broken line upon a straight line equals the projection, upon the same straight line, of the straight line which joins the extremities of the broken line. It follows, therefore, that the projection of the perimeter of any closed polygon upon any given line is zero.

If θ_1, θ_2, θ_3, θ_4, and θ_5 be the angles that PQ, QR, RS, ST, and PT respectively make with the line AB, then the projection of the broken line upon AB may also be expressed thus:

$$\begin{aligned}\text{proj. } PQRST \text{ upon } AB &= MN + NK + KL + LH = MH \\ &= PQ\cos\theta_1 + QR\cos\theta_2 + RS\cos\theta_3 + ST\cos\theta_4 \\ &= PT\cos\theta_5.\end{aligned}$$

The projections of two parallel segments of equal length upon any given line in space are equal. It therefore follows that:

(1) The projection of a segment of a line upon any straight

line in space equals the product of its length by the cosine of the angle between the two lines.

(2) The projection of any broken line in space upon any straight line equals the projection, upon the same line, of the straight line which joins the extremities of the broken line.

EXERCISES

1. Two lines of lengths 3 and 7 respectively meet at an angle $\frac{\pi}{3}$; find the projection of each upon the other.

2. The center of an equilateral triangle, of side 5, is joined by a straight line to a vertex; find the projection of this joining line upon each side of the triangle.

3. A rectangle has its sides respectively 4 and 6; find their projections upon a diagonal.

4. Find the length of the projection of each edge of a cube upon a chosen diagonal.

5. A given line AB makes an angle of 30° with the line MN, and BC is perpendicular to AB and of length 15; find the projection of BC upon MN.

Solve this problem if the given angle be α instead of 30°.

6. Two lines in space, of length a and b respectively, make an angle ω with each other; find the projection of b upon a line that is perpendicular to a.

7. Project the perimeter of a square upon one of its diagonals.

CHAPTER II

GEOMETRIC CONCEPTIONS. THE POINT

I. COÖRDINATE SYSTEMS

18. Coördinates of a point. Position, like magnitude, is relative, and can be given for a geometric figure only by reference to some fixed geometric figures (planes, lines, or points) which are regarded as known, just as magnitude can be given only by reference to some standard magnitudes which are taken as units of measurement. The position of the city of New York, for example, when given by its latitude and longitude, is referred to the equator and the meridian of Greenwich, — the position of these two lines being known, that of New York is also known. So also the position of Baltimore may be given by its distance and direction from Washington; while a particular point in a room may be located by its distances from the floor and two adjacent walls.

If, as in the last illustration, a point is to be fixed in *space*, then *three* magnitudes must be known, referring to *three* fixed positions. If, on the other hand, the point is on a known surface, as New York or Baltimore on the surface of the earth, then only *two* magnitudes need be known, referring to two fixed positions on that surface; while if the point is on a known line, only *one* magnitude, referring to one fixed position on that line, is needed to fix its position.

These various magnitudes which serve to fix the position of a point, — in space, on a surface, or on a line, — are called the **coördinates** of the point.

19. Analytic Geometry. Coördinates may be represented by algebraic numbers; the relations of the various points, and the properties of the various geometric figures which are formed by those points, can be studied through the corresponding relations of these algebraic numbers, or coördinates, expressed in the form of algebraic equations. This fact is the basis of analytic, or algebraic, geometry, the main object of which is the study of geometric properties by algebraic methods.

Analytic geometry may be conveniently divided into two parts: **Plane Analytic Geometry**, which treats only of figures in a given plane surface; and **Solid Analytic Geometry**, which treats of space figures, and includes *Plane Analytic Geometry* as a special case. The plane analytic geometry, being the simpler, will be studied first, in Part I of this book, and Part II will be devoted to the study of the solid analytic geometry. In this first part of the subject it will therefore be understood that the work is restricted to a given plane surface.

Two systems of coördinates will be used, the **Cartesian** and the **Polar**. They are explained in the next few articles.

20. Positive and negative coördinates. If a point lies in a given directed straight line, its position with reference to a fixed point of that line is completely determined by *one* coördinate. *E.g.*, let $X'OX$ be a given directed straight line, and let distances from O toward X be regarded as positive, then distances from O toward X' are negative. A point P,

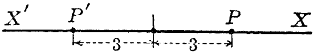


Fig. 6.

in this line and 3 units from O toward X may be designated by $+3$, where the sign $+$ gives the *direction* of the point, and the number 3 its distance, from O. Under these circumstances the point P' lying 3 units on the other side of O would be designated by -3.

In the same way there corresponds to every real number, positive or negative, a definite point of this directed straight line; the numbers are called the **coördinates** of the points; and O, from which the distances are measured, is called the **origin** of coördinates.

21. Cartesian coördinates of points in a plane. Suppose two directed straight lines $X'OX$ and $Y'OY$ are given, fixed in the plane and intersecting in the point O. These two given lines are called the **coördinate axes**, $X'OX$ being the x-axis, and $Y'OY$ being the y-axis; their point of intersection O is the **origin** of coördinates. Any other two lines, parallel respectively to these fixed lines, and at known distances from them, will intersect in one and but one point P, whose position is thus definitely fixed. If these lines through P meet the axes in M and L respectively, then the directed distances LP and MP, *measured parallel respectively to the axes*, are the Cartesian coördinates of the point P. The distance LP, or its equal OM, is the **abscissa** of P, and is usually represented by x, while MP, or its equal OL, is the **ordinate** of P, and is usually represented by y. The point P is designated by the symbol (x, y),—often written $P \equiv (x, y)$, — the abscissa always being written first, then a comma, then the ordinate, and both letters being

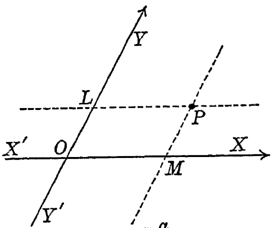


Fig. 7.a

inclosed in a parenthesis. Thus the point (4, 5) is the point for which $OM = 4$ and $MP = 5$; while the point $(-3, 2)$ has $OM = -3$ and $MP = 2$.

22. Rectangular coördinates. The simplest and most common form of Cartesian coördinate axes is that in which the angle XOY is a positive right angle; the abscissa (x) of a point is, in this case, its *perpendicular* distance from the y-axis, and its ordinate (y) is its perpendicular distance from the x-axis. This way of locating the points of a plane is known as the **rectangular system of coördinates.** The axes divide the entire plane into four parts called **quadrants,** which are usually designated as first (I), second (II), third (III), and fourth (IV), in the order of rotation from the positive end of the x-axis toward the positive end of the y-axis, as indicated in the accompanying figure.

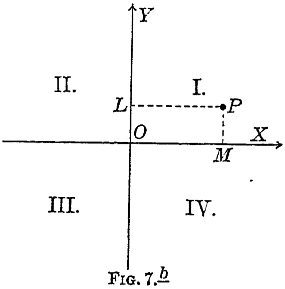


Fig. 7.b

These quadrants are distinguished by the *signs* of the coördinates of the points lying within them, thus:

in quadrant I the abscissa (x) is +, the ordinate (y) is +;
in quadrant II the abscissa (x) is −, the ordinate (y) is +;
in quadrant III the abscissa (x) is −, the ordinate (y) is −;
in quadrant IV the abscissa (x) is +, the ordinate (y) is −.

Four points having numerically the same coördinates, but lying one in each quadrant, are symmetrical in pairs with regard to the origin, even though the axes are not at right angles; if, however, the axes are rectangular, then these points are symmetrical in pairs, not merely with regard to the origin as before, but also with regard to the axes, and they are severally equidistant from the origin. Because of this greater symmetry rectangular coördinates have many advantages over an oblique system.

In the following pages rectangular coördinates will always be understood unless the contrary is expressly stated.

EXERCISES

1. Plot accurately the points: (1, 7), ($^-4$, $^-5$),* (0, 3), and ($^-3$, 0).

2. Plot accurately, as vertices of a triangle, the points: (1, 3), (2, 7), and ($^-4$, $^-4$). Find by measurement the lengths of the sides, and the coördinates of the middle point of each side.

3. Construct the two lines passing through the points (2, $^-7$) and ($^-2$, 7), and (2, 7) and ($^-2$, $^-7$), respectively. What is their point of intersection? Find the coördinates of the middle point of each line.

4. If the ordinate of a point is 0, where is the point? if its abscissa is 0? if its abscissa is equal to its ordinate? if its abscissa and ordinate are numerically equal but of opposite signs?

5. Express each of the conditions of Ex. 4 by means of an equation.

6. The base of an equilateral triangle, whose side is 5 inches, coincides with the x-axis; its middle point is at the origin; what are the coördinates of the vertices? If the axes are chosen so as to coincide with two sides of this triangle, respectively, what are the coördinates of the vertices?

7. A square whose side is 5 inches has its diagonals lying upon the coördinate axes; find the coördinates of its vertices. If a diagonal and an adjacent side are chosen as axes, what are the coördinates of the vertices? of the middle points of the sides? of the center?

8. Find, by similar triangles, the coördinates of the point which bisects the line joining the points (2, 7) and (4, 4).

9. Show that the distance from the origin to the point (a, b) is $\sqrt{a^2 + b^2}$. How far from the origin is the point $(a, {}^-b)$? $({}^-a, b)$? $({}^-a, {}^-b)$? (cf. Art. 22.)

10. Prove, by similar triangles, that the points: (2, 3), (1, $^-3$), and (3, 9) lie on the same straight line.

11. Solve exercises 1 to 4 and 10 if the coördinate axes make an angle of 60°. Also if this angle be 45°.

---

* These minus signs are written high merely to indicate that they are signs of *quality* and not of *operation*.

23. Polar coördinates. If a fixed point O is given in a fixed directed straight line OR, then the position of any point P of the plane will be fully determined by its distance

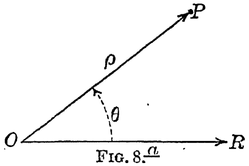


FIG. 8.a

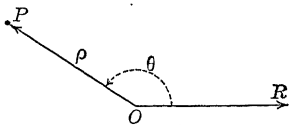


FIG. 8.b

$OP = \rho$ from the fixed point, and by the angle θ which the line OP makes with the fixed line.

The fixed line OR is called the **initial line** or **polar axis,** the fixed point O the **pole** of the system, and the **polar coördinates** of the point P are the **radius vector** ρ and the **directional** or **vectorial angle** θ. The usual rule of signs applies to the vectorial angle θ, and the radius vector is positive if measured from O along the terminal side of the angle θ. The point P is designated by the symbol (ρ, θ).

From what has just been said it is clear that *one* pair of polar coördinates (*i.e.*, one value of ρ and one of θ) serve to determine one, and but one, point of the plane. On the other hand, if θ is restricted to values lying between 0 and 2π, then any given point may be designated by *four* different pairs of coördinates.

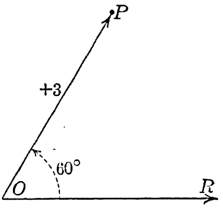


FIG. 9.a

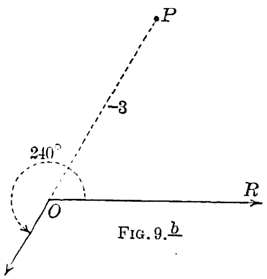


FIG. 9.b

E.g., the polar coördinates (3, 60°) determine the position of the point P for which $OP = 3$ and makes an angle of 60°

IISc Lib B'lore
516.3 M98
13357

with the initial line *OR*, but the same point may be given equally well by the pairs of coördinates: $(^-3, 240°)$, $(3, {}^-300°)$, and $(^-3, {}^-120°)$; and so in general.

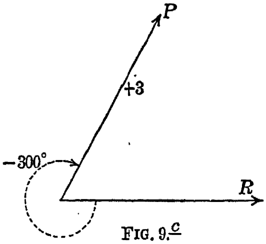


FIG. 9.c

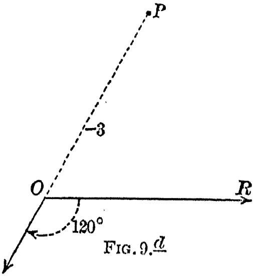


FIG. 9.d

EXERCISES

1. Plot accurately the following points: $(2, 20°)$, $\left(2, \frac{\pi}{9}\right)$, $\left({}^-7, \frac{\pi}{2}\right)$, $\left(4\pi, \frac{\pi^{(r)}}{3}\right)$, $(2, 14\pi°)$, $(^-1, {}^-180°)$, $(7, {}^-45°)$, $(^-7, 135°)$, $\left(5, \frac{3\pi}{4}\right)$, $\left(0, \frac{\pi}{3}\right)$, $\left(0, \frac{^-\pi}{3}\right)$, $(6, 0°)$, and $(^-6, 0°)$.

2. Construct the triangles whose vertices are: $\left(2, \frac{\pi}{8}\right)$, $\left(3, \frac{3\pi}{4}\right)$, $\left(1, \frac{5\pi}{4}\right)$; find by measurement the lengths of the sides and the coördinates of their middle points.

3. The base of an equilateral triangle, whose side is 5 inches, is taken as the polar axis, with the vertex as pole; find the coördinates of the other two vertices.

4. Write three other pairs of coördinates for each of the points $\left(2, \frac{\pi}{4}\right)$; $(^-3, 75°)$; $(5, 0°)$; $(0, 60°)$.

5. Where is the point whose radius vector is 7? whose radius vector is -7? whose vectorial angle is 25°? whose vectorial angle is $0^{(r)}$? whose vectorial angle is $-180°$?

6. Express each of the conditions of Ex. 6 by means of an equation.

7. What is the direction of the line through the points $\left(3, \frac{\pi}{4}\right)$ and $\left(3, \frac{3\pi}{4}\right)$?

24. Notation. In the following pages, to secure uniformity and in accordance with Art. 6, a variable point will be designated by P, and its coördinates by (x, y) or (ρ, θ). If several variable points are under consideration at the same time, they will be designated by $P, P', P'', P''', \cdots$, and their coördinates by (x, y), (x', y'), (x'', y''), (x''', y'''), $\cdots$, or by (ρ, θ), (ρ', θ'), (ρ'', θ''), (ρ''', θ'''), $\cdots$. Fixed points will be designated by $P_1, P_2, \cdots$, and their coördinates by (x_1, y_1), (x_2, y_2), $\cdots$, or by (ρ_1, θ_1), (ρ_2, θ_2), $\cdots$.

II. ELEMENTARY APPLICATIONS

25. The methods of representing a point in a plane that have been adopted in the previous articles lead at once to several easy applications, such as finding the distance between two points, the area of a triangle, etc. The *form* of the results will depend upon the particular system of coördinates chosen, but the *method* is the same in each case. Here, as in the more difficult problems that arise later, to gain the full advantage of the analytic method the student should freely use geometric constructions to guide his algebraic work, but he should, at the same time, see clearly that the method is essentially algebraic.

26. Distance between two points.

(1) *Polar coördinates.* Let OR be the initial line,* O the pole, and let $P_1 \equiv (\rho_1, \theta_1)$ and $P_2 \equiv (\rho_2, \theta_2)$ be the two given

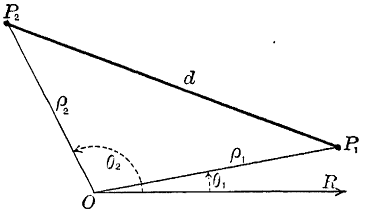


FIG. 10.$\underline{a}$

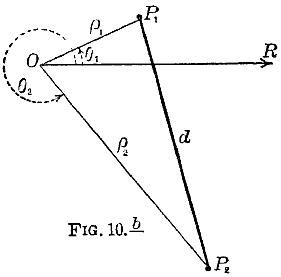


FIG. 10.$\underline{b}$

---

* The demonstration applies to each figure.

fixed points. It is required to find the distance $P_1P_2 = d$ in terms of the given constants ρ_1, ρ_2, θ_1, and θ_2. In the triangle OP_1P_2 (cf. Art. 16)

$$\overline{P_1P_2}^2 = \overline{OP_1}^2 + \overline{OP_2}^2 - 2 \cdot OP_1 \cdot OP_2 \cdot \cos P_1OP_2,$$

i.e., $$d^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos(\theta_2 - \theta_1),$$

hence $$\mathbf{d = \sqrt{\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos(\theta_2 - \theta_1)}} \quad . \quad . \quad . \quad . \quad [1]$$

(2) *Cartesian coördinates; axes not rectangular.* Let OX and OY be the coördinate axes, meeting at an angle $XOY = \omega$,* and let $P_1 \equiv (x_1, y_1)$ and $P_2 \equiv (x_2, y_2)$ be the two given points; it is required to find the distance $P_1P_2 = d$ in terms of x_1, x_2, y_1, y_2, and ω.

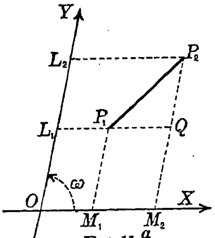


FIG. 11.^{a}

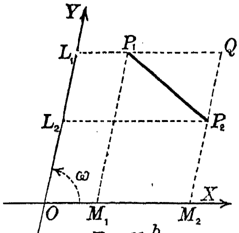


FIG. 11.^{b}

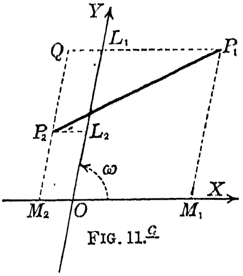


FIG. 11.^{c}

Construction: Extend the abscissa L_1P_1 of the point P_1 to meet the ordinate M_2P_2 of the point P_2, in Q; then in the triangle P_1QP_2 (cf. Art 16)

$$\overline{P_1P_2}^2 = \overline{P_1Q}^2 + \overline{QP_2}^2 - 2 \cdot P_1Q \cdot QP_2 \cdot \cos P_1QP_2, \text{ Fig. } 11^a,$$

$$\overline{P_1P_2}^2 = \overline{P_1Q}^2 + \overline{P_2Q}^2 - 2 \cdot P_1Q \cdot P_2Q \cdot \cos P_1QP_2, \text{ Fig. } 11^b,$$

$$\overline{P_1P_2}^2 = \overline{QP_1}^2 + \overline{P_2Q}^2 - 2 \cdot QP_1 \cdot P_2Q \cdot \cos P_1QP_2, \text{ Fig. } 11^c;$$

which gives, for each figure,

$$\mathbf{d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + 2(x_1 - x_2)(y_1 - y_2)\cos\omega}.}\dagger$$

---

* The demonstration applies to each figure.

† By examining other possible constructions the student should assure himself of the generality of this formula.

(3) *Rectangular coördinates.* If $\omega = \frac{\pi}{2}$, *i.e.* if the coördinate axes are rectangular, then $\cos \omega = 0$, and the formula for the distance between the two given points becomes

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \quad . \quad . \quad . \quad [2]$$

Since either of the two points may be named P_1, this formula may be expressed in words thus: *In rectangular coördinates, the square of the distance between two given points is the square of the difference between their abscissas plus the square of the difference between their ordinates.*

27. Slope of a line. By the **slope** of a line is meant the tangent of the angle which the line makes with the positive end of the x-axis.*

From this definition it at once follows that the slope m of the line joining the two points $P_1 \equiv (x_1, y_1)$ and $P_2 \equiv (x_2, y_2)$, the axes being rectangular, is $m = \frac{QP_2}{P_1Q}$; that is,

$$m = \frac{y_2 - y_1}{x_2 - x_1}. \quad . \quad . \quad . \quad [3]$$

EXERCISES

1. Find the distances between the points (1, 3), (2, 7), and (−4, −4), taken in pairs.

2. Find the distances for the points of Ex. 1, if the axes are oblique with $\omega = 60°$.

3. Prove that the points (−2, −1), (1, 0), (4, 3), and (1, 2) are the vertices of a parallelogram.

4. Find the distance between the points $(a + b,\ c + a)$ and $(c + a,\ b + c)$; also between (a, b) and $(-a, -b)$.

5. Find the distances between the points (2, 30°), $\left(3, \frac{3\pi}{4}\right)$, and $\left(1, \frac{5\pi}{4}\right)$, taken in pairs.

---

* The slope of a roof or of a hill has the same meaning. Thus if the slope of a hill (to the horizontal) is $\frac{3}{100}$, it rises 3 feet vertical in 100 feet horizontal.

6. Prove that the points $(0, 0^r)$, $\left(3, \frac{\pi}{2}\right)$, and $\left(3, \frac{\pi}{6}\right)$ form an equilateral triangle.

7. One end of a line whose length is 13 is at the point $(^-4, 8)$, the ordinate of the other end is 3; what is its abscissa?

8. Express by an equation the fact that the point $P \equiv (x, y)$ is at the distance 3 from the point $(^-2, 3)$; from the point $(0, 0)$.

9. Express by an equation the fact that the point $P \equiv (x, y)$ is equidistant from the points $(^-2, 3)$ and $(7, 5)$.

10. Find the slopes of the lines which join the following pairs of points: $(3, 8)$ and $(^-1, 4)$; $(2, ^-3)$ and $(7, 9)$; $(1, ^-4)$ and $(^-3, 5)$; $(4, ^-2)$ and $(^-2, ^-1)$.

28. One great advantage of the analytic method of solving problems lies in the fact that the analytic results which are obtained from the simplest arrangement of the geometric figure with reference to the coördinate axes are, from the very nature of the method, equally true for all other arrangements. Thus formulas [1], [2], and [3] can be most readily obtained if the points are all taken in quadrant I, *i.e.*, with their coördinates all positive; but because of the convention adopted concerning the signs as essential parts of the coördinates, these formulas remain true for all possible positions of P_1 and P_2. By drawing the figures and making the proofs when P_1 and P_2 are taken in various other positions, the student should assure himself of the generality of formulas [1], [2], and [3] of articles 26 and 27.

29. The area of a triangle.

1. *Rectangular coördinates.* Given a triangle with the vertices $P_1 \equiv (x_1, y_1)$, $P_2 \equiv (x_2, y_2)$, and $P_3 \equiv (x_3, y_3)$; to find its area in terms of x_1, x_2, x_3, y_1, y_2, and y_3. Draw the ordinates M_1P_1, M_2P_2, and M_3P_3,—in the second figure extend M_1P_1 and M_3P_3 to meet a line through P_2 parallel to the x-axis. If Δ represents the area of the triangle in the first figure, then:

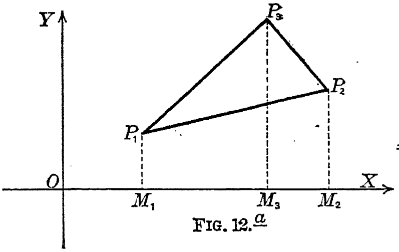


FIG. 12.a

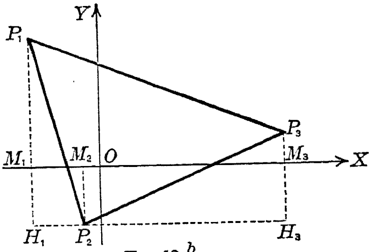


FIG. 12.b

$$\Delta = P_1M_1M_3P_3 + P_3M_3M_2P_2 - P_1M_1M_2P_2,$$

but $P_1M_1M_3P_3 = \frac{1}{2}(M_1P_1 + M_3P_3)\cdot M_1M_3 = \frac{1}{2}(y_1+y_3)(x_3-x_1)$,

and $P_3M_3M_2P_2 = \frac{1}{2}(M_3P_3 + M_2P_2)\cdot M_3M_2 = \frac{1}{2}(y_3+y_2)(x_2-x_3)$,

and $P_1M_1M_2P_2 = \frac{1}{2}(M_1P_1 + M_2P_2)\cdot M_1M_2 = \frac{1}{2}(y_1+y_2)(x_2-x_1)$.

$$\therefore\ \Delta = \tfrac{1}{2}\{(y_1+y_3)(x_3-x_1) + (y_2+y_3)(x_2-x_3) - (y_1+y_2)(x_2-x_1)\}$$

$$= \tfrac{1}{2}\{(y_1+y_2)(x_1-x_2) + (y_2+y_3)(x_2-x_3) + (y_3+y_1)(x_3-x_1)\} \quad . \;.\; . \quad [4]$$

This may also be written in the form

$$\Delta = \tfrac{1}{2}\{x_1(y_2-y_3) + x_2(y_3-y_1) + x_3(y_1-y_2\}^{*} \quad . \;.\; . \quad [4\,a]$$

So also if Δ_1 represents the area of the triangle in the second figure, then

$$\Delta_1 = P_1H_1H_3P_3 - P_1H_1P_2 - P_2H_3P_3$$

$$= \tfrac{1}{2}\{(H_1P_1 + H_3P_3)\cdot M_1M_3 - H_1P_1\cdot H_1P_2 - H_3P_3\cdot P_2H_3\},$$

$$= \tfrac{1}{2}\{(y_1-y_2+y_3-y_2)(x_3-x_1) - (y_1-y_2)(x_2-x_1) - (y_3-y_2)(x_3-x_2)\},$$ [x_1, x_2, and y_2 being negative]

$$= \tfrac{1}{2}\{x_1(y_2-y_3) + x_2(y_3-y_1) + x_3(y_1-y_2)\},^{*}$$ as above [4 a].

---

* In the determinant notation this may be written

$$\text{area of the triangle} = \tfrac{1}{2}\begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix}.$$

If, instead of rectangular coördinate axes, oblique axes making an angle $XOY=\omega$ had been used, it would have been necessary merely to multiply the second members in the results just found by $\sin\omega$ in order to express the areas of the triangles.

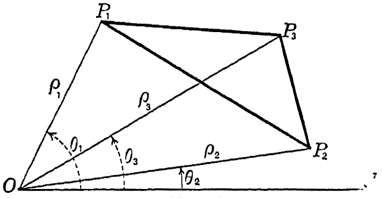


FIG. 13.

2. *Polar coördinates.* Let the vertices of the triangle be $P_1\equiv(\rho_1,\ \theta_1)$, $P_2\equiv(\rho_2,\ \theta_2)$, and $P_3\equiv(\rho_3,\ \theta_3)$; to find its area Δ in terms of ρ_1, ρ_2, ρ_3, θ_1, θ_2, and θ_3.

Manifestly, $\Delta=OP_2P_3+OP_3P_1-OP_2P_1$,

but $OP_2P_3=\frac{1}{2}\rho_2\rho_3\sin(\theta_3-\theta_2)$, $OP_3P_1=\frac{1}{2}\rho_3\rho_1\sin(\theta_1-\theta_3)$, and $OP_2P_1=\frac{1}{2}\rho_2\rho_1\sin(\theta_1-\theta_2)$.

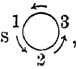
$$\therefore\ \Delta=\tfrac{1}{2}\{\rho_2\rho_3\sin(\theta_3-\theta_2)+\rho_3\rho_1\sin(\theta_1-\theta_3)-\rho_2\rho_1\sin(\theta_1-\theta_2)\},$$

which may also be written

$$\Delta=\tfrac{1}{2}\{\rho_1\rho_2\sin(\theta_2-\theta_1)+\rho_2\rho_3\sin(\theta_3-\theta_2)+\rho_3\rho_1\sin(\theta_1-\theta_3)\}.\quad .\ .\ .\ .\ [5]$$

The symmetry* in formulas [4], [4*a*], and [5] should be carefully noted; it may be remarked also, that in the application of these formulas to numerical examples, the resulting areas will be positive or negative according to the relative order in which the vertices are named.

---

* This kind of symmetry is known as *cyclic* (or circular) symmetry. If the numbers 1, 2, and 3 be arranged thus , then the subscripts in the first term (in [4*a*] say) begin with 1 and follow the arrow heads around the circle (*i.e.* their order is 1, 2, 3), those of the second term begin with 2 and follow the arrow heads (their order is 2, 3, 1), and those of the third term begin with 3 and follow the arrow heads.

EXERCISES

1. Find the areas of the following triangles: (1) vertices at the points (3, 5), (4, 2), and (1, 3); (2) vertices at the points (7, 3), (4, 6), and (3, −2); (3) vertices at the points (11, 9), (6, −2), and (−5, 3).

Solve without using the formula, and then verify by substituting in the formula.

2. Prove that the area of the triangle whose vertices are at the points (2, 3), (5, 4), and (−4, 1) is zero, and hence that these points all lie on the same straight line.

3. Do the points (2, 3), (1, −3), and (3, 9) lie on one straight line? (cf. Ex. 10, p. 28.)

4. Do the points (7, 30°), (0, 0°), and (−11, 210°) lie on one straight line? Solve this by showing that the area of the triangle is zero, and then verify by plotting the figure.

5. Find the area of the triangle $\left(\pi, \frac{\pi^{(r)}}{2}\right)$, $\left(2\pi, \frac{\pi^{(r)}}{2}\right)$, and $\left(-\pi, \frac{2\pi^{(r)}}{3}\right)$.

6. Derive formula [4] when P_1 is in quadrant II, P_2 in quadrant III, and P_3 in quadrant IV.

7. Find the area of the first two triangles in Ex. 1 if the axes make an angle of 60° with each other.

30. To find the coördinates of the point which divides in a given ratio the straight line from one given point to another. Let $P_1 \equiv (x_1, y_1)$ and $P_2 \equiv (x_2, y_2)$ be the two given points, $P_3 \equiv (x_3, y_3)$ the required point, and let the ratio of the parts into which P_3 divides P_1P_2 be $m_1 : m_2$; *i.e.*, let $P_1P_3 : P_3P_2 = m_1 : m_2$. Draw the ordinates M_1P_1,

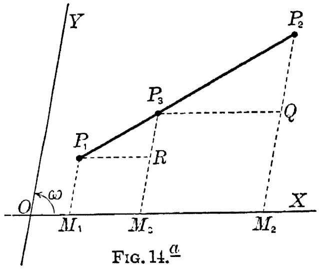


FIG. 14.a

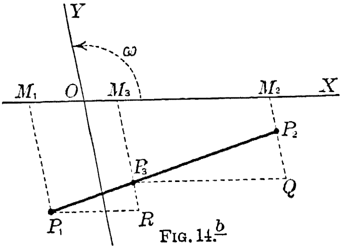


FIG. 14.b

M_2P_2, M_3P_3, and, through P_1 and P_3, draw lines parallel to OX, meeting M_3P_3 and M_2P_2 in R and Q respectively.

To find $OM_3 = x_3$ and $M_3P_3 = y_3$ in terms of x_1, x_2, y_1, y_2, m_1, and m_2.

The triangles P_1RP_3 and P_3QP_2 are similar;

therefore
$$\frac{P_1R}{P_3Q} = \frac{RP_3}{QP_2} = \frac{P_1P_3}{P_3P_2}.$$

But
$$\frac{P_1P_3}{P_3P_2} = \frac{m_1}{m_2},$$

and
$$P_1R = x_3 - x_1,\ P_3Q = x_2 - x_3,$$
$$RP_3 = y_3 - y_1,\ QP_2 = y_2 - y_3.$$

[In Fig. 14 (*b*), x_1, y_1, y_2, and y_3 are negative.]

therefore
$$\frac{x_3 - x_1}{x_2 - x_3} = \frac{y_3 - y_1}{y_2 - y_3} = \frac{m_1}{m_2};$$

whence
$$x_3 = \frac{m_1x_2 + m_2x_1}{m_1 + m_2} \text{ and } y_3 = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}. \quad . \quad . \quad [6]$$

The above reasoning applies equally well whatever the value of ω (the angle made by the coördinate axes), hence formulas [6] hold whether the axes be rectangular or oblique.

Formulas [6] were obtained on the implied hypothesis that P_3 lies *between* P_1 and P_2; *i.e.*, that P_3 is an *internal* point of division. If P_3 is taken in the line P_1P_2 produced, and not between P_1 and P_2, it still forms, with P_1 and P_2, two segments P_1P_3 and P_3P_2, and P_3 may be so taken that, numerically, the ratio of $P_1P_3 : P_3P_2$ may have any real value whatever; but the *sign* of this ratio is negative when P_3 is not between P_1 and P_2, for, in that case, the segments P_1P_3 and P_3P_2 have opposite directions. Hence, to find the coördinates of that point which divides a line *externally* into segments whose *numerical* ratio is $m_1 : m_2$, it is only necessary to prefix the minus sign to either one of the two numbers m_1 or m_2 in formulas [6]. These formulas then become

$$x_3 = \frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \quad y_3 = \frac{m_1y_2 - m_2y_1}{m_1 - m_2}. \quad . \quad . \quad . \qquad [7]$$

COR. If P_3 be the middle point of P_1P_2, then $m_1 = m_2$, and formulas [6] become

$$x_3 = \frac{x_1+x_2}{2}, \qquad y_3 = \frac{y_1+y_2}{2}; \quad . \quad . \quad . \qquad [8]$$

i.e., the abscissa of the middle point of the line joining two given points is half the sum of the abscissas of those points, and the ordinate is half the sum of their ordinates.

The remarks in Art. 28 are well illustrated by formulas [4] to [8].

EXERCISES

1. By means of an appropriate figure, derive formulas [7] independently of [6].

2. The point $P_3 \equiv (2, 3)$ is one third of the distance from the point $P_1 \equiv (-1, 4)$ to the point $P_2 \equiv (x_2, y_2)$; to find the coördinates of P_2.

Here P_1 and P_3 are given, with $x_1 = -1$, $y_1 = 4$, $x_3 = 2$, $y_3 = 3$, also $m_1 = 1$, and $m_2 = 2$; therefore, from [6],

$$2 = \frac{x_2 + 2(-1)}{1 + 2}, \text{ and } 3 = \frac{y_2 + 2(4)}{1 + 2},$$

which give $x_2 = 8$ and $y_2 = 1$; therefore the required point P_2 is (8, 1).

3. Find the points of trisection of the line joining (1, −2) to (3, 4).

4. Find the point which divides the line from (1, 3) to (−2, 4) externally into segments whose numerical ratio is 3 : 4.

Here $x_1 = 1$, $y_1 = 3$, $x_2 = -2$, $y_2 = 4$, $m_1 = 3$, and $m_2 = 4$, but the point of division being an external one, the two segments are oppositely directed; therefore one of the numbers 3 or 4, say 4, must have the minus sign prefixed to it. Substituting these values in [6],

$$x_3 = \frac{3(1) - 4(-2)}{3 - 4} = -11, \text{ and } y_3 = \frac{3(3) - 4(4)}{3 - 4} = 7;$$

the required division point is, therefore, $P_3 \equiv (-11, 7)$.

The same result would have been obtained had $m_1 = 3$, instead of $m_2 = 4$, been given the minus sign; or, again, formulas [7] could have been employed to solve this problem.

5. Solve Ex. 4 directly from a figure, without using either [6] or [7].

6. Find the points which divide the line from (1, 5) to (2, 7) internally and externally into segments which are in the ratio 2 : 3.

7. A line AB is produced to C, so that $BC = \frac{1}{2}AB$; if the points A and B have the coördinates (5, 6) and (7, 2), respectively, what are the coördinates of C?

8. Prove, by means of Art. 30, that the median lines of a triangle meet in a point, which is for each median the point of trisection nearest the side of the triangle.

31. Fundamental problems of analytic geometry. The elementary applications already considered have indicated how algebra may be applied to the solution of geometric problems. Points in a plane have been identified with pairs of numbers, — the coördinates of those points, — and it has been seen that definite relations between such points correspond to definite relations between their coördinates.

It will be found also that the relation between points, which consists in their lying on a definite curve, corresponds to the relation between their coördinates, which consists in their satisfying a definite equation. From this fact arise the two fundamental problems of analytic geometry:

I. *Given an equation, to find the corresponding geometric curve, or locus.*

II. *Given a geometric curve, to find the corresponding equation.*

When this relation between a curve and its equation has been studied, then a third problem arises:

III. *To find the properties of the curve from those of its equation.*

The first two problems will be treated in the two succeeding chapters, while the remaining chapters of Part I will be concerned chiefly with the third problem. In this application of analytic methods, however, only algebraic equations of the first and second degrees will for the most part be considered. In Chapter XIII is given a brief study of other important equations and curves.

EXAMPLES ON CHAPTER II

1. Find the area of the quadrilateral whose vertices are the points $(1, 0)$, $(3, \frac{5}{2})$, $(-1, 16)$, and $(-4, 2)$. Draw the figure.

2. Find the lengths of the sides and the altitude of the isosceles triangle $(1, 5)$, $(5, 1)$, $(-9, -9)$. Find the area by two different methods, so that the results will each be a check on the other.

3. Find the coördinates of the point that divides the line from $(2, 3)$ to $(-1, -6)$ in the ratio $3:4$; in the ratio $2:-3$; in the ratio $3:-2$. Draw each figure.

4. One extremity of a straight line is at the point $(-3, 4)$, and the line is divided by the point $(1, 6)$ in the ratio $2:3$; find the other extremity of the line.

5. The line from $(-6, -2)$ to $(3, -1)$ is divided in the ratio $4:5$; find the distance of the point of division from the point $(-4, 6)$.

6. Find the area and also the perimeter of the triangle whose vertices are the points $(3, 60^\circ)$, $(5, 120^\circ)$, and $(8, 30^\circ)$.

7. Show analytically that the figure formed by joining the middle points of the sides of any quadrilateral is a parallelogram.

8. Show that the points $(1, 3)$, $(2, \sqrt{6})$, and $(2, -\sqrt{6})$ are equidistant from the origin.

9. Show that the points $(1, 1)$, $(-1, -1)$, and $(-\sqrt{3}, -\sqrt{3})$ form an equilateral triangle. Find the slopes of its sides.

10. Prove analytically that the diagonals of a rectangle are equal.

11. Show that the points $(0, -1)$, $(2, 1)$, $(0, 3)$, and $(-2, 1)$ are the vertices of a square.

12. Express by an equation that the point (h, k) is equidistant from $(-1, 1)$ and $(1, 2)$; from $(1, 2)$ and $(1, -2)$. Then show that the point $(\frac{3}{4}, 0)$ is equidistant from $(-1, 1)$, $(1, 2)$, and $(1, -2)$.

13. Prove analytically that the middle point of the hypotenuse of a right triangle is equidistant from the three vertices.

14. Three vertices of a parallelogram are $(1, 2)$, $(-5, -3)$, and $(7, -6)$; what is the fourth vertex?

15. The center of gravity of a triangle is at the point in which the medians intersect. Find the center of gravity of the triangle whose vertices are $(2, 3)$, $(4, -5)$, and $(3, -6)$. (cf. Ex. 8, p. 40.)

16. The line from (x_1, y_1) to (x_2, y_2) is divided into five equal parts; find the points of division.

17. Prove analytically that the two straight lines which join the middle points of the opposite sides of a quadrilateral mutually bisect each other.

18. Prove that $(1, 5)$ is on the line joining the points $(0, 2)$ and $(2, 8)$, and is equidistant from them.

19. If the angle between the axes is 30°, find the perimeter of the triangle whose vertices are $(2, 2)$, $(-7, -1)$, and $(-1, 1)$. Plot the figure.

20. Show analytically that the line joining the middle points of two sides of a triangle is half the length of the third side.

21. A point is 7 units distant from the origin and is equidistant from the points $(2, 1)$ and $(-2, -1)$; find its coördinates.

22. Prove that the points $(a, b + c)$, $(b, c + a)$, and $(c, a + b)$ lie on the same straight line. (cf. Ex. 2, p. 37.)

CHAPTER III

THE LOCUS OF AN EQUATION

32. The locus of an equation. A pair of numbers x, y is represented geometrically by a point in a plane. If these two numbers (x, y) are variables, but connected by an equation, then this equation can, in general, be satisfied by an infinite number of pairs of values of x and y, and each pair may be represented by a point. These points will not, however, be scattered indiscriminately over the plane, but will all lie in a definite curve, whose form depends only upon the nature of the equation under consideration; and this curve will contain no points except those whose coördinates are pairs of values which when substituted for x and y, satisfy the given equation. This curve is called the **locus** or **graph** of the equation; and the first fundamental problem of analytic geometry is to find, for a given equation, its graph or locus.

33. Illustrative examples: Cartesian coördinates.

(1) *Given the equation* $x + 5 = 0$, *to find its locus.* This equation is satisfied by the pairs of values $x_1 = -5$, $y_1 = 2$; $x_2 = -5$, $y_2 = 3$; $x_3 = -5$, $y_3 = -2$; etc., that is, by every pair of values for which $x = -5$. Such points as

$$P_1 \equiv (x_1, y_1) \equiv (-5, 2),$$
$$P_2 \equiv (x_2, y_2) \equiv (-5, 3),$$
$$P_3 \equiv (x_3, y_3) \equiv (-5, -3), \text{ etc.,}$$

all lie on the line MN, parallel to the y-axis, and at the distance 5 on the negative side of it, — this line extending indefinitely in both directions. Moreover, each point of MN has for its abscissa -5, hence the coördinates of each of its points satisfy the equation $x + 5 = 0$. In the chosen system of coördinates, the line MN is called the locus of this equation.

Similarly, the equation $x - 5 = 0$ is satisfied by any pair of values of which x is 5, such as (5, 2), (5, 3), (5, 4), etc.; all the corresponding points lie on a straight line $M'N'$, parallel to the y-axis, at the distance 5 from it, and on its positive side; *i.e.*, $M'N'$ is the locus of the equation $x - 5 = 0$.

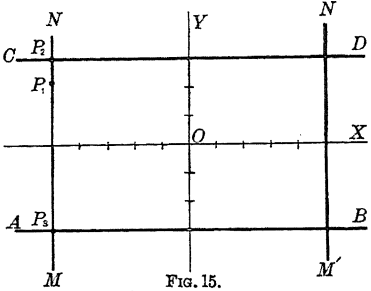


FIG. 15.

(2) *Given the equations* $y \pm 3 = 0$, *to find their loci.* By the same reasoning as in (1) it may be shown that the locus of the equation $y + 3 = 0$ is the straight line AB, parallel to the x-axis, situated at the distance 3 from it, and on its negative side. Also that the locus of the equation $y - 3 = 0$ is CD, a line parallel to the x-axis, at the distance 3 from it, and on its positive side.

More generally, it is evident that *in Cartesian coördinates (rectangular or oblique), an equation of the first degree, and containing but one variable, represents a straight line parallel to one of the coördinate axes.*

(3) *Given the equation* $3x - 2y + 12 = 0$, *to find its locus.* In this equation *both* the variables appear. By assigning any definite value to either one of the variables, and solving the equation for the other, a pair of values that will satisfy the equation is obtained. Thus the following pairs of values are found:

$$\begin{array}{ll}
x_1 = 0,\ y_1 = 6 & x_5 = -1,\ y_5 = 4\tfrac{1}{2} \\
x_2 = 1,\ y_2 = 7\tfrac{1}{2} & x_6 = -2,\ y_6 = 3 \\
x_3 = 2,\ y_3 = 9 & x_7 = -3,\ y_7 = 1\tfrac{1}{2} \\
x_4 = 3,\ y_4 = 10\tfrac{1}{2} & x_8 = -4,\ y_8 = 0 \\
\cdots\cdots & \cdots\cdots \\
x = +\infty,\ y = +\infty & x = -\infty,\ y = -\infty
\end{array}$$

Plotting the corresponding points $P_1, P_2, P_3, P_4 \cdots$, where $P_1 \equiv (x_1, y_1) \equiv (0, 6)$, $P_2 \equiv (x_2, y_2) \equiv (1, 7\tfrac{1}{2})$, etc., they are all found to lie on the straight line EF, which is the locus of the equation $3x - 2y + 12 = 0$.

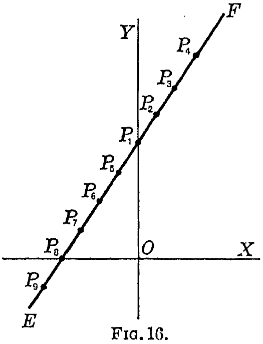


FIG. 16.

In Chap. V, it will be shown that, in Cartesian coördinates, an equation of the first degree in two variables always represents a straight line.

(4) *Given the equation $y^2 = 4x$, to find its locus.* This equation is satisfied by each of the following pairs of values, found as in (3) above:

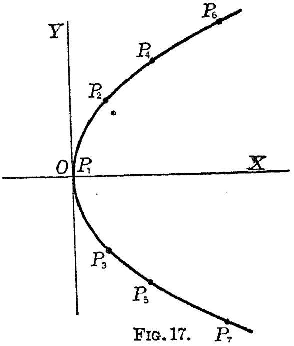


FIG. 17.

$x_1 = 0,\ y_1 = 0$
$x_2 = 1,\ y_2 = +2$
$x_3 = 1,\ y_3 = -2$
$x_4 = 2,\ y_4 = 2\sqrt{2} = 2.8$, approximately
$x_5 = 2,\ y_5 = -2\sqrt{2} = -2.8$, approximately
$x_6 = 4,\ y_6 = +4$
. . . .
$x = +\infty,\ y = \pm\infty$

and for any negative value of x the corresponding value of y is imaginary.

The corresponding points are:

$P_1 \equiv (0, 0)$, $P_2 \equiv (1, 2)$, $P_3 \equiv (1, -2)$, etc.

All these points are found to lie on the curve as plotted in Fig. 17. This curve is called a *parabola*, and will be studied in a later chapter.

The parabola is one of the curves obtained by the intersection of a circular cone and a plane. (cf. Appendix, Note D.) It will be shown in Chap. XII that in Cartesian coördinates, the locus of any algebraic equation in two variables and of the second degree is a "conic section."

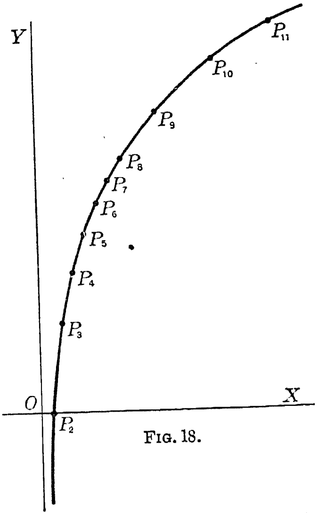


FIG. 18.

(5) *Given the equation, $y = 25 \log x$, to find its locus.* A table of logarithms shows that this equation is satisfied by the following pairs of values:

$x_1 = 0,\ y_1 = -\infty$
$x_2 = 1,\ y_2 = 0$
$x_3 = 2,\ y_3 = 7.5$
$x_4 = 3,\ y_4 = 11.9$
$x_5 = 4,\ y_5 = 15$
$x_6 = 5,\ y_6 = 17.5$

$x_7 = 6,\ y_7 = 19.4$
$x_8 = 7,\ y_8 = 21.1$
$x_9 = 10,\ y_9 = 25$
$x_{10} = 15,\ y_{10} = 29.4$
$x_{11} = 20,\ y_{11} = 32.5$
etc. etc.

The corresponding points are:

$P_1 \equiv (0, -\infty)$, $P_2 \equiv (1, 0)$, $P_3 \equiv (2, 7.5)$, etc.; and the locus of the above equation is approximately given by the curve drawn through these points as shown in Fig. 18.

(6) *Given the equation* $y = \tan x$; *to find its locus.* By means of a table of "natural" tangents it is seen that this equation is satisfied by the following pairs of values of x and y:

Degrees	Radians	y
$x_1 = 0$	$= 0.00$	$y_1 = 0$
$x_2 = 10$	$= 0.17$	$y_2 = 0.18$
$x_3 = 20$	$= 0.35$	$y_3 = 0.36$
$x_4 = 30$	$= 0.52$	$y_4 = 0.58$
$x_5 = 40$	$= 0.70$	$y_5 = 0.84$
$x_6 = 50$	$= 0.87$	$y_6 = 1.19$
$x_7 = 60$	$= 1.05$	$y_7 = 1.73$
$x_8 = 70$	$= 1.22$	$y_8 = 2.75$
$x_9 = 80$	$= 1.40$	$y_9 = 5.67$
$x_{10} = 90$	$= 1.57$	$y_{10} = \infty$
$x_{11} = -10$	$= -0.17$	$y_{11} = -0.18$
$x_{12} = -20$	$= -0.35$	$y_{12} = -0.36$
$x_{13} = -30$	$= -0.52$	$y_{13} = -0.58$
etc.	etc.	etc.

The corresponding points are:

$$P_1 \equiv (0, 0),\ P_2 \equiv (0.17, 0.18),\ P_3 \equiv (0.35, 0.36), \text{ etc.,}$$

and the locus is approximately as shown in Fig. 19.

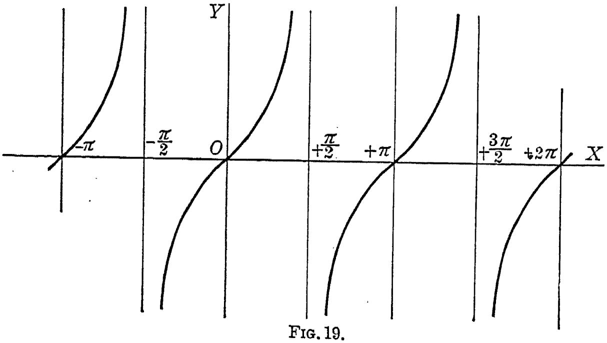


Fig. 19.

34. Loci by polar coördinates. Analogous results are obtained for a system of polar coördinates, as will be best seen from an example. *Given the equation* $\rho = 4 \cos \theta$, *to find its locus.*

This equation is satisfied by the following pairs of values, found as in Art. 33 (3) and (4):

$\theta_1 = 0$	$\rho_1 = 4$
$\theta_2 = 30^\circ$	$\rho_2 = 2\sqrt{3} = 3.46+$
$\theta_3 = 60^\circ$	$\rho_3 = 2$
$\theta_4 = 45^\circ$	$\rho_4 = 2\sqrt{2} = 2.8+$
$\theta_5 = 90^\circ$	$\rho_5 = 0$
$\theta_6 = -30^\circ$	$\rho_6 = 3.46+$
$\theta_7 = -60^\circ$	$\rho_7 = 2$
$\theta_8 = -45^\circ$	$\rho_8 = 2.8+$
$\theta_9 = -90^\circ$	$\rho_9 = 0$
etc.	etc.

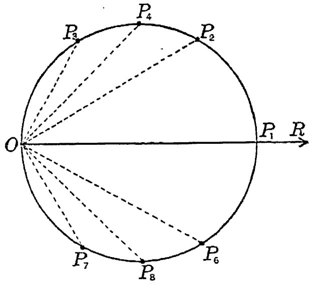


FIG. 20.

The corresponding points are:

$P_1 \equiv (4, 0^\circ)$; $P_2 \equiv (3.46+, 30^\circ)$; $P_3 \equiv (2, 60^\circ)$; $P_4 \equiv (2.8+, 45^\circ)$; $P_5 \equiv P_9 \equiv$ the pole $O \equiv (0, \pm 90^\circ)$; $P_6 \equiv (3.46+, -30^\circ)$; $P_7 \equiv (2, -60^\circ)$; etc.

All these points are found to lie on the circumference of a circle whose radius is 2, the pole being on the circumference, and the polar axis being a diameter. This circle is the locus of the equation $\rho = 4 \cos \theta$.

EXERCISES

Plot the loci of the following equations:

1. $x = 0$.
2. $y = 0$.
3. $bx = 0$.
4. $3x = 7$.
5. $2y + 5 = 0$.
6. $x + y = 0$.
7. $x^2 + y^2 = 4$.
8. $x + y = 4$.
9. $x - y = 0$.
10. $x^2 - y^2 = 4$.
11. $2x^2 + y^2 = 4$.
12. $v = 32t$.
13. $t^2 + s^2 = 9$.
14. $u^2 + v = 0$.
15. $s = 16t^2$.
16. $\frac{x}{2} + \frac{y}{3} = 1$.
17. $\rho = 3$.
18. $\rho \cos(\theta - 40^\circ) = 5$.
19. $y = -x^3$.

35. The locus of an equation. By the process illustrated above, of constructing a curve from its equation, the first conception of a locus is obtained, viz.:

(1) *The locus of an equation containing two variables is the line, or set of lines, which contains all the points whose coördinates satisfy the given equation, and which contains no other points.* It is the *place where* all the points, and

only those points, are found whose coördinates satisfy the given equation.

A second conception of the locus of an equation comes directly from this one, for the line or set of lines may be regarded as the *path traced* by a point which moves along it. The path of the moving point is determined by the condition that its coördinates for every position through which it passes must satisfy the given equation. Thus the line EF (the locus of eq. (3), Art. 33) may be regarded as the path traced by the point P, which moves so that its coördinates (x, y) always satisfy the equation

$$3x - 2y + 12 = 0.$$

Thus arises a second conception of a locus, viz.:

(2) *The locus of an equation is the path traced by a point which moves so that its coördinates always satisfy the given equation.*

In either conception of a locus, the essential condition that a point shall lie on the locus of a given equation is, that *the coördinates of the point when substituted respectively for the variables of the equation, shall satisfy the equation;* and in order that a curve may be the locus of an equation, it is necessary that *there be no other points than those of this curve whose coördinates satisfy the equation.*

36. Classification of loci. The form of a locus depends upon the nature of its equation; the curve may therefore be classified according to its equation, an algebraic curve being one whose equation is algebraic, and a transcendental curve one whose equation is transcendental. In particular, the **degree** *of an algebraic curve* is defined to be the same as the degree of its equation. The following pages are concerned chiefly with algebraic curves of the first and second degrees.

37. Construction of loci. Discussion of equations. The process of constructing a locus by plotting separate points, and then connecting them by a smooth curve, is only approximate, and is long and tedious. It may often be shortened by a consideration of the peculiarities of the given equation, such as symmetry, the limiting values of the variables for which both are real, etc. Such considerations will often show the general form and limitations of the curve; and, taken together, they constitute a *discussion of the equation.*

The points where a locus crosses the coördinate axes are almost always useful; in drawing the curve, they are given by their distances from the origin along the respective axes. These distances are called the **intercepts** of the curve.

The following examples may serve to illustrate these conceptions:

(1) *Discussion of the equation* $3x - 2y + 12 = 0$ [see (3) Art. 33].

Intercepts: if $x = 0$, then $y = 6$; hence the y-intercept is 6 (see Fig. 16); if $y = 0$, then $x = 4$; hence the x-intercept is 4.

The equation may be written: $x = \frac{2}{3}y - 4$, which shows that as y increases continuously from 0 to ∞, x increases continuously from -4 to ∞; therefore the locus passes from the point P_8 through the point P_1, and then recedes indefinitely from both axes in the first quadrant. Written as above, the equation also shows that as y decreases from 0 to $-\infty$, x also decreases from -4 to $-\infty$; therefore the locus passes from P_8 into the third quadrant, receding again indefinitely from both axes. Since for every value of y, x takes but one value (*i.e.*, each value of y corresponds to but one point on the curve), therefore the locus consists of a single branch. The *proof* that the locus of any first-degree equation, in two variables, is a straight line is given in Chap. V.

(2) *Discussion of the equation* $y^2 = 4x$. [See (4) Art. 33.]

Intercepts (see Fig. 17): if $x = 0$, then $y = 0$, and if $y = 0$, then $x = 0$;

hence the locus cuts each axis in one point only, and that point is the origin. The equation may be written in the form $y = \pm\sqrt{4x}$, which shows that if x be negative y is imaginary; hence there is no point of this locus on the negative side of the y-axis.

Again: for each positive value of x there are two real values of y, numerically equal, but opposite in sign; hence this locus passes through the origin, lies wholly in the first and fourth quadrants, and is symmetrical with regard to the x-axis.

The equation shows also that x may have any positive value, however great, and that y increases when x increases; these facts show that the locus recedes indefinitely from both axes, — that it is an open curve of one branch. It is called a parabola and has the form shown in Fig. 17.

(3) *Discussion of the equation* $x^2 + y^2 = a^2$.

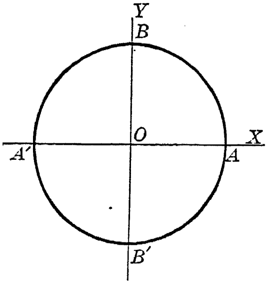


FIG. 21.

Intercepts: if $x = 0$, then $y = \pm a$, and if $y = 0$, then $x = \pm a$; hence for each axis there are *two* intercepts, each of length a, and on opposite sides of the origin; *i.e.*, four positions of the tracing point are: $A \equiv (a, 0)$, $A' \equiv (^-a, 0)$, $B \equiv (0, a)$, and $B' \equiv (0, ^-a)$.

This equation may also be written

$$y = \pm\sqrt{a^2 - x^2},$$

which shows that every value of x gives two corresponding values of y which are numerically equal, but of opposite sign; the locus is, therefore, symmetrical with regard to the x-axis. It also shows that, corresponding to any value of x numerically greater than a, y is imaginary; the tracing point, therefore, does not move further from the y-axis than $\pm a$, *i.e.*, further than the points A and A'. Moreover, as x increases from 0 to a, y remains real and changes gradually from $+a$ to 0, or from $-a$ to 0; *i.e.*, the tracing point moves continuously from B to A, or from B' to A.

Again, if x decreases from 0 to $-a$, y remains real and changes continuously from $+a$ to 0, or from $-a$ to 0; *i.e.*, the tracing point moves continuously from B to A' or from B' to A'.

Similarly, the equation may be written $x = \pm\sqrt{a^2 - y^2}$, which shows that the curve is also symmetrical with regard to the y-axis, and that the tracing point does not move farther than $\pm a$ from the x-axis.

From these facts it follows that this locus is a closed curve of only one branch. That it is a circle of radius a, with its center at the origin, will be shown in Chap. VII.

(4) *Discussion of the equation* $y^2 = (x-2)(x-3)(x-4)$.

Intercepts: if $x = 0$, then y is imaginary; if $y = 0$, then $x = 2, 3$, or 4; hence the locus crosses the x-axes at the three points: $A \equiv (2, 0)$, $B \equiv (3, 0)$, and $C \equiv (4, 0)$, and it does not cut the y-axis at all. Moreover, since y is imaginary if x is negative, the locus lies wholly on the positive side of the y-axis.

This locus is symmetrical with regard to the x-axis; it has no point nearer to the y-axis than A; between A and B it consists of a closed branch; and it has no real points between B and C, but is again real beyond C. The entire locus consists, then, of a closed oval, and of an open branch which recedes indefinitely from both axes, see Fig. 22.

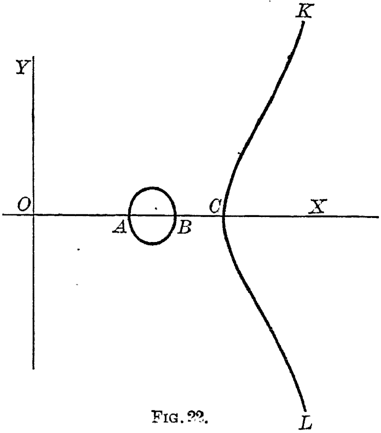


Fig. 22.

(5) *Discussion of the equation* $y = \tan x$. This equation has already been examined in (6) Art. 33, but in practice it may be much more simply plotted by the following method:

Describe a circle with unit radius; draw the diameter AOC, and the lines OB_1, OB_2, OB_3, $\cdots$, meeting the tangent AT in the points T_1, T_2, T_3, $\cdots$; then the tangent of the angle AOB_1 is $M_1B_1 : OM_1 = AT_1 : OA$ (Art. 14), and, since $OA = 1$, its value is graphically represented by AT_1. So also

$$\tan AOB_2 = M_2B_2 : OM_2 = AT_2 : OA = AT_2 : 1,$$

and may be graphically represented by AT_2. In the same way, AT_3, AT_4, AT_6, $\cdots$ are the tangents of the angles AOB_3, AOB_4, AOB_6, $\cdots$. Again, since angles at the center of a circle are proportional to the arcs intercepted by their sides, AT_1, AT_2, $\cdots$ may be said to be the tangents of the arcs AB_1, AB_2, $\cdots$; *i.e.*, $AT_1 = \tan AB_1$, $AT_2 = \tan AB_2$, $\cdots$. Therefore the coördinates of the points $P_1 \equiv (AB_1, AT_1)$, $P_2 \equiv (AB_2, AT_2)$, $\cdots$ satisfy the given equation, and if a sufficient number of points, whose coördinates are thus determined, be plotted, they will all lie on a curve like that in Fig. 19.

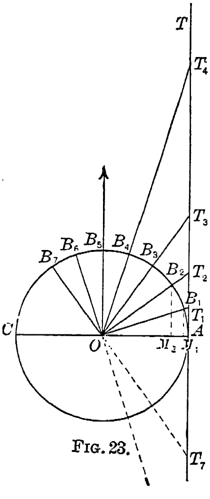


Fig. 23.

From what has just been said it is clear that $y = 0$ if $x = 0$, hence the curve goes through the origin; when x increases continuously from 0 to $\frac{\pi}{2}$, y increases continuously from 0 to ∞, but when x increases *through* $\frac{\pi}{2}$, y passes suddenly from $+\infty$ to $-\infty$, and the curve is *discontinuous* for that value of x. So also when x increases continuously from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$, y increases continuously from $-\infty$ through 0 to $+\infty$, and is again discontinuous for $x = \frac{3\pi}{2}$. The locus consists of an infinite number of such infinite, but continuous branches, separated by the points of discontinuity for which $x = \pm\frac{\pi}{2}$, $x = \pm\frac{3\pi}{2}$, $x = \pm\frac{5\pi}{2}$, ….

The other trigonometric functions, $y = \sin x$, $y = \sec x$, etc., can all be plotted by a method analogous to that above.

EXERCISES

Construct and discuss the loci of the following equations:

1. $\frac{x^2}{4} - \frac{y^2}{9} = 1.$

2. $\frac{x^2}{4} + \frac{y^2}{9} = 1.$

3. $y = \sec x.$

4. $x^2 - y^2 = a^2.$

5. $x^2 - y^2 = 0.$

6. $4x^2 - y^2 = 0.$

7. $v = \sin u.$

8. $x^2 + y^2 = 0.$

9. $\frac{y-1}{y-2} = 5^{\frac{1}{x-1}}$ (cf. Ex. 8, p. 8.)

38. The locus of an equation remains unchanged: (α) by any transposition of the terms of the equation; and (β) by multiplying both members of the equation by any finite constant.

(α) If in any equation the terms are transposed from one member to the other in any way whatever, the locus of the equation is not changed thereby; for the coördinates of all the points which satisfied the equation in its original form, and only those coördinates, satisfy it after the transpositions are made. [See Art. 35 (1).]

(β) If both members of an equation are multiplied by any finite constant k, its locus is not changed thereby. For if the terms of the equation, after the multiplication has been performed, are all transposed to the first member, that member may be written as the product of the constant k and a factor containing the variables. This product will vanish if, and only if, its second factor vanishes; but this factor will vanish if, and only if, the variables which it contains are the coördinates of points on the locus of the original equation. Hence the coördinates of all points on the locus of the original equation, and only those coördinates, satisfy the equation *after* it has been multiplied by k; hence the locus remains unchanged if its equation is multiplied by a finite constant.

39. Points of intersection of two loci. Since the points of intersection of two loci are points on *each* locus, therefore the coördinates of these points must satisfy each of the two equations; moreover, the coördinates of no other points can satisfy *both* equations. Hence, to find the coördinates of the points of intersection of two curves, it is only necessary to regard their equations as simultaneous and solve for the coördinates.

E.g., Find the coördinates of the points of intersection, P_1 and P_2, of the loci of $x - 2y = 0$, and $y^2 = x$. The point of intersection $P_1 \equiv (x_1, y_1)$ is on *both* curves,

$$\therefore\ x_1 - 2y_1 = 0, \text{ and } y_1^2 = x_1.\text{*}$$

Solving these two equations,

$$x_1 = 0, \text{ or } 4, \text{ and } y_1 = 0, \text{ or } 2;$$

i.e., $P_1 \equiv (4, 2)$ and $P_2 \equiv (0, 0)$ are two points, the coördinates of which satisfy each of the two given equations; therefore they are the points of intersection of the loci of these equations.

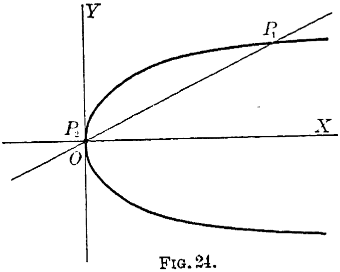


FIG. 24.

EXERCISES

Find the points of intersection of the following pairs of curves:

1. $\begin{cases} 7x - 11y + 1 = 0, \\ x + y - 2 = 0. \end{cases}$

2. $\begin{cases} x + y = 3, \\ x - y = 3. \end{cases}$

---

* If x and y are regarded as the coördinates of the point of intersection, the subscripts may be omitted here.

3. $\begin{cases} y = 3x + 2, \\ x^2 + y^2 = 4. \end{cases}$

4. $\begin{cases} 2y - 5x = 0, \\ x^2 - y^2 = 5. \end{cases}$

5. $\begin{cases} x^2 + y^2 = 9, \\ x^2 + 6xy + y^2 = 0. \end{cases}$

6. $\begin{cases} x^2 + y^2 = a^2, \\ 3x + y + a = 0. \end{cases}$

7. $\begin{cases} y^2 = 4px, \\ y - x = 0. \end{cases}$

8. $\begin{cases} x + y = 2a, \\ b^2x^2 + a^2y^2 = a^2b^2. \end{cases}$

9. $\begin{cases} x^2 + y^2 = 16, \\ x^2 - 2y^2 = 1. \end{cases}$

10. $\begin{cases} y^2 = 4x, \\ y - x = 3. \end{cases}$

11. $\begin{cases} \rho = 2\cos\theta, \\ \rho\cos\theta = 4. \end{cases}$

12. $\begin{cases} \rho = 9\cos(45° - \theta), \\ \rho\cos\left(\frac{\pi}{2} + \theta\right) = 1. \end{cases}$

13. Trace carefully the above loci; by measurement, find the coördinates of the points in which each pair intersect; and compare these results with those already obtained by computation.

40. Product of two or more equations. *Given two or more equations with their second members zero;* the product of their first members, equated to zero, has for its locus the combined loci of the given equations.*

This follows at once from the fundamental relation between an equation and its locus (see Art. 35 (1)), for the new equation is satisfied by the coördinates of those points which make one of its factors zero, but it is satisfied by the coördinates of no other points; *i.e.*, this new equation is satisfied by the coördinates of points that lie on one or another of the loci of the given equations.

The following example illustrates this principle in the case of two given equations.

Let the given equations be:

$$x + y = 0 \quad . \quad . \quad . \quad (1) \quad \text{and} \quad x - y = 0 \quad . \quad . \quad . \quad (2)$$

---

* If equations whose second members are *not* zero are multiplied together, member by member, the resulting equation is not satisfied by any points of the loci of the given equations except those in which they intersect each other; the new equation therefore represents a locus through the points of intersection of the loci of the given equations.

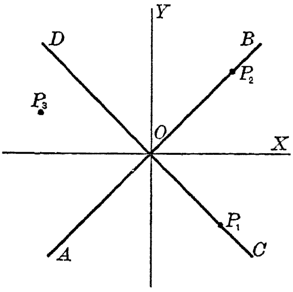


FIG. 25.

Equation (1) represents the straight line CD, and equation (2) the line AB, — bisecting respectively the angles between the axes. It is to be shown that the equation

$$(x + y)(x - y) = 0 \quad . \quad . \quad . \quad (3)$$

(or, what is the same, $x^2 - y^2 = 0$), formed from equations (1) and (2), has for its locus both these lines.

Proof. If $P_1 \equiv (x_1, y_1)$ is any point on CD, then its coördinates satisfy equation (1), hence $x_1 + y_1 = 0$, and therefore $(x_1 + y_1)(x_1 - y_1) = 0$; which shows that P_1 is a point of the locus of equation (3). But since P_1 was *any* point of CD, therefore the coördinates of every point on CD satisfy equation (3); *i.e.*, all points of CD belong to the locus of equation (3).

In the same way it is shown that AB belongs to the locus of equation (3).

Moreover, if $P_3 \equiv (x_3, y_3)$ be any point not on AB nor on CD, then $x_3 + y_3 \neq 0$, and $x_3 - y_3 \neq 0$, hence

$$(x_3 + y_3)(x_3 - y_3) \neq 0;$$

i.e., P_3 does not belong to the locus of equation (3).

Hence the locus of equation (3) contains the loci of equations (1) and (2), but contains no other points.

The above theorem may be stated briefly thus: if u, v, w, etc., be any functions of two variables, then the equation $uvw \cdot \cdots = 0$ has for its locus the combined loci of the equations $u = 0$, $v = 0$, $w = 0$, etc.

NOTE. When possible, factoring the first member of an equation, whose second member is zero, simplifies the work of finding the locus of the given equation.

EXERCISES

What loci are represented by the following equations?

1. $xy = 0.$ **2.** $\frac{x^2}{4} - \frac{y^2}{9} = 0.$ **3.** $3x^2 + 2xy - 7x = 0.$

4. $5xy^2 - 2x^2y = 0.$ **5.** $x^2 - 2x + 1 = 0.$ **6.** $(x^2 + y^2 - 4)(y^2 - 4x) = 0.$

41. Locus represented by the sum of two equations. Suppose the equations

$$2y - x = 0 \quad . \quad . \quad . \quad (1), \text{ and } \quad y^2 - x = 0 \quad . \quad . \quad . \quad (2)$$

are given. Their loci are respectively AB and DP_2P_1C (Art. 39), and it is required to find the locus of their sum; *i.e.*, of $2y - x + y^2 - x = 0$, or, what is the same thing, of

$$y^2 + 2y - 2x = 0 \quad . \quad . \quad . \quad (3)$$

The locus of this last equation passes through all the points in which AB and DP_2P_4C intersect each other. For let $P_1 \equiv (x_1, y_1)$ be one of these points, then since P_1 lies on AB, its coördinates satisfy equation (1); *i.e.*,

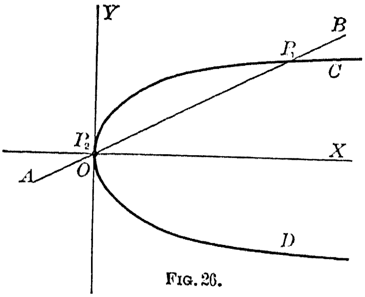


Fig. 26.

$$2y_1 - x_1 = 0; \quad . \quad . \quad . \quad (4)$$

and since P_1 lies on DP_2P_1C, its coördinates satisfy equations (2); *i.e.*,

$$y_1^2 - x_1 = 0; \quad . \quad . \quad . \quad (5)$$

therefore, by adding equations (4) and (5),

$$y_1^2 + 2y_1 - 2x_1 = 0. \quad . \quad . \quad . \quad (6)$$

This last equation proves (Art. 35 (1)) that $P_1 \equiv (x_1, y_1)$ is on the locus of equation (3); *i.e.*, the locus of equation (3) passes through $P_1 \equiv (x_1, y_1)$.

Similar reasoning would show that the locus of equation

(3) passes through every other point in which the loci of equations (1) and (2) intersect each other.

In precisely the same way it may be proved generally that *the locus of the sum of two equations passes through all the points in which the loci of the two given equations intersect each other.*

If either of the given equations (1) or (2) had been multiplied by any constant factor before adding, the above reasoning would still have led to the same conclusion; in fact, this theorem may be briefly, and more generally, stated thus: *if u and v are any functions of the two variables x and y, and k is any constant, then the locus of*

$$u + kv = 0$$

passes through every point of intersection of the loci of

$$u = 0 \text{ and } v = 0.$$

For, let the locus of the equation $u = 0$ be the curve ABC, the locus of $v = 0$ be the curve DEF, and let $P_1 \equiv (x_1, y_1)$ be *any* one of the points in which these curves intersect each other.

Then the equation

$$u + kv = 0$$

is satisfied by the coördinates of the point $P_1 \equiv (x_1, y_1)$, because if these coördinates be substituted for x and y in the functions u and v they must make both these functions separately equal to zero. Therefore the locus of $u + kv = 0$ passes through every point in which the loci of $u = 0$ and $v = 0$ intersect each other.

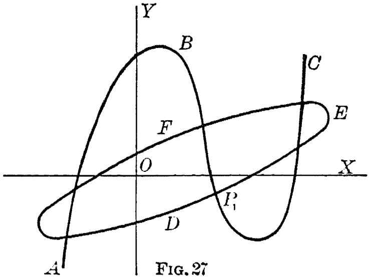


FIG. 27

EXERCISES

1. Verify Art. 41 by first finding the coördinates of the points of intersection of the loci of equations (1) and (2), and then substituting these coördinates in equation (3).

2. Find the equation of a curve that passes through all the points in which the following pairs of curves intersect:

$$(\alpha)\ \left\{\begin{array}{l} x^2+y^2=9, \\ x^2+2\,x+y^2=0. \end{array}\right\} \qquad (\beta)\ \left\{\begin{array}{l} y=\sin x, \\ y=2\cos x. \end{array}\right\}$$

3. Find the equation of a curve through all the points common to the following pairs of curves:

$$(\alpha)\ \left\{\begin{array}{l} x^2=4\,y, \\ y^2=4\,x. \end{array}\right\} \qquad (\beta)\ \left\{\begin{array}{l} \rho=2\cos\theta, \\ \rho\cos\theta=\frac{1}{4}. \end{array}\right\}$$

NOTE. It is to be observed that the method given in Art. 39, for finding the point of intersection of two curves, is an application of the theorem of Art. 41. For the process of solving two simultaneous equations, at least one of which involves two variables, consists in combining them in such a way as to obtain two simple equations, each involving only one variable. Now each of these simple equations represents an elementary locus, — one or more straight lines parallel to the axes, if the coördinates are Cartesian; circles about the pole, or straight lines through the pole, if the coördinates are polar, — and these elementary loci determine, *i.e.*, pass through, the points of intersection of the original loci. To determine the points of intersection, then, of two loci, the original loci are replaced by simpler ones passing through the same common points. *E.g.*, the points of intersection of the loci of Art. 39,

$$2\,y-x=0 \quad . \quad . \quad . \quad (1), \quad \text{and} \quad y^2=x, \quad . \quad . \quad . \quad (2)$$

are given by the equations

$$(y^2-x)-(2\,y-x)=0 \quad \text{and} \quad (2\,y-x)^2-4\,(y^2-x)=0,$$

that is, by $\qquad y^2-2\,y=0, \quad \text{and} \quad x^2-4\,x=0,$

which may be written

$$y(y-2)=0 \quad . \quad . \quad . \quad (3), \qquad x(x-4)=0. \quad . \quad . \quad . \quad (4)$$

But the locus of equation (3) is a pair of straight lines parallel to the x-axis, and the locus of equation (4) is a pair of straight lines parallel to the y-axis; and these loci have the same points of intersection as the loci (1) and (2).

EXAMPLES ON CHAPTER III

1. Are the points (3, 9), (4, 6), and (5, 5) on the locus of $3x+2y=25$?

2. Is the point $\left(\frac{a}{2}, \frac{a}{3}\right)$ on the locus of $4x^2+9y^2=2a^2$?

3. The ordinate of a certain point on the locus of $x^2+y^2=25$ is 4; what is its abscissa? What is the ordinate if the abscissa is a^2?

Find by the method of Art. 39 where the following loci cut the axes of x and y.

4. $y=(x-2)(x-3)$.

5. $16x^2+9y^2=144$.

6. $x^2+6x+y^2=4y+3$.

Find by the method of Art. 39 where the following loci cut the polar axis (or initial line).

7. $\rho=4\sin^2\theta$.

8. $\rho^2=a^2\cos 2\theta$.

9. The two loci $\frac{x^2}{4}-\frac{y^2}{9}=1$, $\frac{x^2}{4}+\frac{y^2}{9}=1$ intersect in four points; find the lengths of the sides and of the diagonals of the quadrilateral formed by these points.

10. A triangle is formed by the points of intersection of the loci of $x+y=a$, $x-2y=4a$, and $y-x+7a=0$. Find its area.

11. Find the distance between the points of intersection of the curves $3x-2y+12=0$, and $x^2+y^2=9$.

12. Does the locus of $y^2=4x$ intersect the locus of $2x+3y+2=0$?

13. Does the locus of $x^2-4y+4=0$ cut the locus of $x^2+y^2=1$?

14. For what values of m will the curves $x^2+y^2=9$ and $y=6x+m$ not intersect? (cf. Art. 9.) Trace these curves.

15. For what value of b will the curves $y^2=4x$ and $y=x+b$ intersect in two distinct points? in two coincident points? in two imaginary points (*i.e.*, not intersect)?

16. Find those two values of c for which the points of intersection of the curves $y=2x+c$ and $x^2+y^2=25$ are coincident.

17. Find the equation of a curve which passes through all the points of intersection of $x^2+y^2=25$ and $y^2=4x$. Test the correctness of the result by finding the coördinates of the points of intersection and substituting them in the equation just found.

let $P \equiv (x, y)$ be *any* other point on the line through P_1 and P_2.

Draw the ordinates M_1P_1, MP, and M_2P_2, and through P_1 draw P_1N parallel to the x-axis, meeting MP in R and M_2P_2 in R_2.

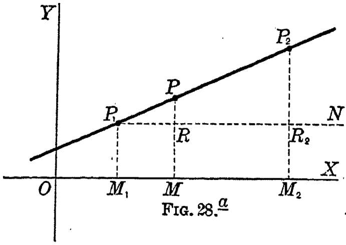


FIG. 28.a

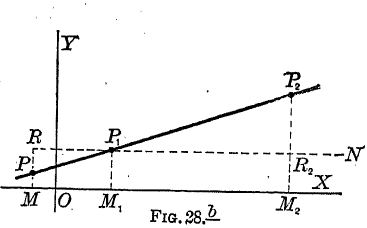


FIG. 28.b

The triangles P_1RP and $P_1R_2P_2$ are similar, hence

$$\frac{RP}{R_2P_2} = \frac{P_1R}{P_1R_2}, \; i.e., \; \frac{MP - M_1P_1}{M_2P_2 - M_1P_1} = \frac{OM - OM_1}{OM_2 - OM_1}.$$

Substituting for MP, OM, M_1P_1, OM_1, etc., their values, this equation becomes

$$\frac{y-2}{5-2} = \frac{x-3}{12-3},$$

which reduces to $\quad 3y - x - 3 = 0. \quad . \quad . \quad . \quad (1)$

This is the required equation of the straight line through P_1 and P_2, because it fulfills both the requirements of the definition [cf. Art. 35 (1)]; *i.e.*, it *is* satisfied by the coördinates of *any* (*i.e.*, of every) point of this line, because x, y are the coördinates of any such point; and it *is not* satisfied by the coördinates of any point which is not on this line, because the corresponding constructions for such a point would not give similar triangles, and hence the proportions which led to this equation would not be true.

That equation (1) is not satisfied by the coördinates of any point not on the line through P_1 and P_2 may also be seen as follows:

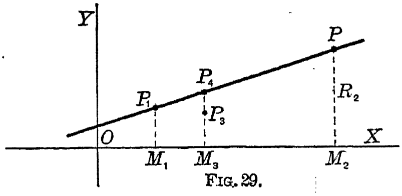


FIG. 29.

let $P_3 \equiv (x_3, y_3)$ be any point not on the line through P_1 and P_2, the ordinate M_3P_3 will meet P_1P_2 in some point $P_4 \equiv (x_4, y_4)$, for which $x_4 = x_3$ but $y_4 \neq y_3$. Since P_4 is *on* the line P_1P_2, its coördinates satisfy equation (1), therefore

$$3\,y_4 - x_4 - 3 = 0,$$

$$\therefore\ 3\,y_3 - x_3 - 3 \neq 0;^{*} \quad [\text{since } x_4 = x_3 \text{ and } y_4 \neq y_3]$$

hence the coördinates of P_3 do not satisfy the equation

$$3\,y - x = 3.$$

44. Equation of straight line passing through given point and in given direction.† Let $P_1 \equiv (5, 4)$ be the given point, let the given line through P_1 make an angle of 30° with the x-axis, and let $P \equiv (x, y)$ be *any* other point on this line.

Draw the ordinates M_1P_1 and MP, and, through P_1, draw P_1R parallel to the x-axis to meet MP in R. Then

$$\tan RP_1P = \frac{RP}{P_1R} = \frac{MP - M_1P_1}{OM - OM_1}.$$

---

* This proof shows clearly that if the coördinates of any point *on* the straight line through P_1 and P_2 are substituted for x and y in equation (1) the first member will be equal to zero; if the coördinates of any point *below* this line are so substituted the first member will be negative; and if the coördinates of any point *above* this line are so substituted the first member will be positive. This line may then be regarded as the boundary which separates that part of the plane for which $3\,y - x - 3$ is negative from the part for which this function is positive. Because of this fact that side of this line on which P_3 lies may be called the *negative side*, and the other the *positive side*.

† See also Art. 53.

Substituting for M_1P_1, MP, OM_1, OM, and angle RP_1P their values, and remembering that $\tan 30° = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3}$, this equation becomes

$$\tfrac{1}{3}\sqrt{3} = \frac{y-4}{x-5};\ \textit{i.e.},\ x - \sqrt{3}\,y - 5 + 4\sqrt{3} = 0.^*$$

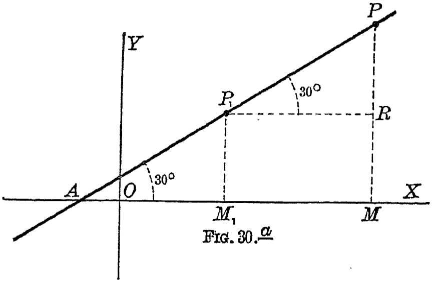


FIG. 30.a

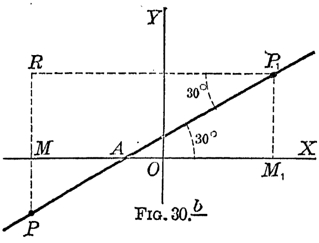


FIG. 30.b

The equation just found is satisfied by the coördinates of any point on the given line, but is not satisfied by the coördinates of any point that is not on this line (cf. Art. 43); hence it is the equation of the line (cf. Art. 35).

45. Equation of a circle; polar coördinates.† In deriving this equation, let polar coördinates be employed, merely for variety, and let the pole be taken on the circumference, with a diameter OA extended for the initial line. Let $P \equiv (\rho, \theta)$ be any point on the circle,‡ and let r be the radius of the circle.

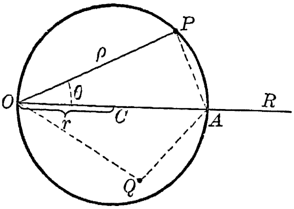


FIG. 31.

Connect P and A by a straight

---

* The positive side of this line is that side on which the origin lies (cf. foot-note, Art. 43).

† See also Art. 98.

‡ Except in plane geometry, the word "circle" is employed by most writers on mathematics to mean "circumference of a circle." It will be so used in this book.

line; then, in triangle AOP, angle OPA is a right angle, $AOP = \theta$, $OP = \rho$, and $OP : OA = \cos\theta$; *i.e.*,

$$\rho : 2r = \cos\theta;$$

hence

$$\rho = 2r\cos\theta. \qquad (1)$$

Equation (1) is satisfied by the polar coördinates of every point on the circle; but is not satisfied by the coördinates of a point Q not on the circle, since angle AQO is not a right angle. Therefore Eq. (1) is the equation of this circle (cf. Art. 35).

EXERCISES

1. Find the equation of the straight line through the two points (1, 7) and (6, 11); through the points ($^-2$, 5) and (3, 8). Which is its positive side of these lines?

2. Find the equation of the straight line through the two points (2, 3) and ($^-2$, $^-3$). Through what other point does this line pass? Does the equation show this fact?

3. Find the equation of the straight line through the point (5, $^-7$), and making an angle of 45° with the x-axis; making the angle $-45°$ with the x-axis.

4. Find the equation of the line through the point ($^-6$, $^-2$), and making the angle 120° with the x-axis.

5. Construct the circle whose equation is $\rho = 10\cos\theta$.

6. With rectangular coördinates, find the equation of the circle of radius 5, which passes through the origin, and has its center on the x-axis. Is its positive side outside or inside?

46. Equation of locus traced by a moving point. In the problems given above, the geometric figure in each case was completely known; and, in obtaining its equation, use was made of the known properties of similar triangles, triangles inscribed in a semicircle, and trigonometric functions. In only a few cases, however, is the curve so completely known; in a large class of important problems, the curve is known merely as the path traced by a point which moves under given conditions or laws. Such a curve, for instance, is the path of a cannon ball, or other projectile, moving under the influence of a known initial force and the force of gravity. Another such curve is that in which iron filings arrange themselves when acted upon by known magnetic forces. The orbits of the planets and other astronomical bodies, acting under the influence of certain centers of force, are important examples of this class of "given loci."

In such problems as these, the method used in Arts. 43 to 45, cannot, in general, be applied. A method that can often be employed, after the construction of an appropriate figure, is:

(1) From the figure, express the known law, under which the point moves, by means of an equation involving geometric magnitudes; this equation may be called the "geometric equation."

(2) Replace each geometric magnitude by its equivalent algebraic value, expressed in terms of the coördinates of the moving point and given constants; then simplify this algebraic equation, and the result is the desired equation of the locus.

47. Equation of a circle: second method. To illustrate this second method of finding the equation of a locus, consider the circle as the path traced by a point which moves so that it is always at a given constant distance from a fixed point. From this definition, find its equation.

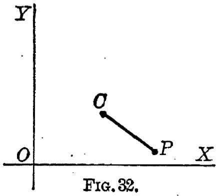


FIG. 32.

Let $C \equiv (3, 2)$ be the given fixed point, and let $P \equiv (x, y)$ be a point that moves so as to be always at the distance $2\frac{1}{2}$ from C. Then

$$CP = \tfrac{5}{2}, \quad . \quad . \quad . \quad \text{[geometric equation]}$$

but $\quad CP = \sqrt{(x-3)^2+(y-2)^2} \qquad$ (Art. 26, [2]),

$\therefore\ \sqrt{(x-3)^2+(y-2)^2} = \frac{5}{2}; \qquad$ [algebraic equation]

i.e., $\quad (x-3)^2+(y-2)^2 = \frac{25}{4};$

hence $\quad 4x^2+4y^2-24x-16y+27=0,$

which is the required equation.

The locus of this equation can now be plotted by the methods of Art. 37, and its form and limitations can be discussed as is there done for other equations.

EXERCISES

1. Find the equation of the path traced by a point which moves so that it is always at the distance 4 from the point (5, 0). Trace the locus.

2. Find the equation of the path traced by a point which moves so that it is always equidistant from the points (-2, 3) and (7, 5) (cf. Ex. 9, p. 34).

3. A line is 3 units long; one end is at the point (-2, 3). Find the locus of the other end (cf. Ex. 8, p. 34).

4. A point moves so as to be always equidistant from the y-axis and from the point (4, 0). Find the equation of its path, and then trace and discuss the locus from its equation.

5. A point moves so that the sum of its distances from the two points $(0, \sqrt{5})$, $(0, -\sqrt{5})$ is always equal to 6. Find the equation of the locus traced by this moving point.

6. A point moves so that the difference of its distances from the two points $(0, \sqrt{5})$, $(0, -\sqrt{5})$ is always equal to 2. Find the equation of the locus traced by this moving point.

48. The conic sections. Of the innumerable loci which may be given by means of the law governing the motion of the generating or tracing point, there is one class of particular importance; and it is to the study of this important class that the following pages will be chiefly devoted. *These curves are traced by a point which moves so that its distance* *from a fixed point always bears a constant ratio to its distance from a fixed straight line.* These curves are called the **Conic Sections**, or more briefly **Conics**, because they can be obtained as the curves of intersection of planes and right circular cones;* in fact, it was in this way that they first became known. The last three examples just given belong to this class, although it is only in No. 4 that this fact is directly stated. These loci are the parabola, the ellipse, and the hyperbola; it will be shown later that they include as special cases the straight line and the circle.† They are of primary importance in astronomy, where it is found that the orbit of a heavenly body is a curve of this kind.

The general equation, which includes all of these curves, will now be derived, and the locus briefly discussed; in a subsequent chapter will be given a detailed study of the properties of these curves in their several special forms.

(*a*) *The equation of the locus.* Let F be the fixed point, — the **focus** of the curve; $D'D$ the fixed line, — the **directrix** of the curve; and e the given ratio, — the **eccentricity** of the curve.

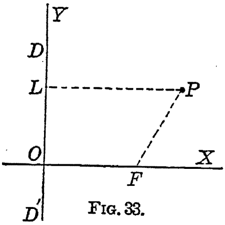


FIG. 33.

The coördinate axes may of course be chosen as is most convenient. Let $D'D$ be the y-axis, and the perpendicular to it through F, *i.e.*, the line OFX, be the x-axis. Let $P \equiv (x, y)$ be any position of the generating point, and let OF, the fixed distance of the focus from the directrix, be denoted by k; then the coördinates of the focus are $(k, 0)$. Connect F and P, and through P draw LP perpendicular to the directrix.

Then $$FP : LP = e, \quad \text{[geometric equation]}$$

---

[*] See Note D, Appendix.

[†] See Note C, Appendix.

but $$FP = \sqrt{(x-k)^2 + y^2} \quad \text{(Art. 26)},$$

and $$LP = x; \quad \text{[algebraic equivalents]}$$

hence $$\sqrt{(x-k)^2 + y^2} = ex;$$

i.e., $$(1-e^2)x^2 + y^2 - 2kx + k^2 = 0, \quad . \quad . \quad . \quad (1)$$

which is the equation of the given locus.

This equation is of the second degree; in a later chapter it will be shown that *every* equation of the second degree between two variables represents a conic section. On this account it is often spoken of as the "second degree curve."

(*b*) *Discussion of equation* (1).

If $x = 0$, then $y = \pm k\sqrt{-1}$, which shows that this curve does not intersect the y-axis as here chosen; *i.e.*, a conic does not intersect its directrix.

If $y = 0$, then $(1-e^2)x^2 - 2kx + k^2 = 0$,

whence $$x = \frac{k}{1+e}, \text{ or } x = \frac{k}{1-e}, \quad . \quad . \quad . \quad (2)$$

i.e., a conic meets the line drawn through the focus and perpendicular to the directrix (the x-axis as here chosen) in two points whose distances from the directrix are $\frac{k}{1+e}$ and $\frac{k}{1-e}$ respectively; these points are called the **vertices** of the conic.

Equation (1) shows that for every value of x, the two corresponding values of y are numerically equal but of opposite signs, hence the conic is symmetrical with regard to the x-axis as here chosen. For this reason the line drawn through the focus of a conic and perpendicular to the directrix is called the **principal axis** of the conic.

The *form* of the locus of equation (1) depends upon the value of the eccentricity (e); if $e = 1$, the conic is called a

parabola; if $e<1$, an **ellipse**; and if $e>1$, an **hyperbola**. Each of these cases will now be separately considered.

(1) *The parabola*, $e=1$. If $e=1$, then $FP:LP=1$, *i.e.*, $FP=LP$ for every position of the tracing point,* hence the curve passes through A, — the point midway between O and F, — but does not again cross the principal axis (cf. also equations (2), above).

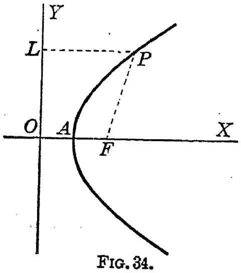


Fig. 34.

Moreover, when $e=1$, equation (1) becomes

$$y^2-2kx+k^2=0,$$

i.e.,

$$y^2=2k\left(x-\frac{k}{2}\right), \quad . \quad . \quad . \quad (3)$$

which is the equation of the parabola, the coördinate axes being the principal axis of the curve and the directrix. Equation (3) shows that there is no point of this parabola for which $x<\frac{k}{2}$, and also that y changes from 0 to $\pm\infty$ when x increases from $\frac{k}{2}$ to ∞; hence the parabola recedes indefinitely from both axes in the first and fourth quadrants. Its form is given in Fig. 34.

(2) *The ellipse*, $e<1$. Equation (1) may be written in the form

$$y^2=(1-e^2)\left(\frac{k}{1-e}-x\right)\left(x-\frac{k}{1+e}\right), \quad . \quad . \quad . \quad (4)†$$

---

* This property enables one to construct any number of points lying on the parabola, thus: with F as center, and any radius not less than $\frac{1}{2}$ OF, describe a circle, then draw a line parallel to OY and at a distance from it equal to the chosen radius; the points in which this line cuts the circle are points on the parabola. Other points can be located in the same way. See also Note B, Appendix.

† Equation (4) enables one to construct any number of points on the

which shows, e being less than 1, that y is imaginary for all values of x except those which satisfy the condition

$$\frac{k}{1+e} \leqq x \leqq \frac{k}{1-e};$$

hence the ellipse lies wholly on the positive side of its directrix, and between two lines which are parallel to the directrix and distant from it $\frac{k}{1+e}$ and $\frac{k}{1-e}$ respectively. Equation (4) shows that as x increases from $\frac{k}{1+e}$ to $\frac{k}{1-e}$, y

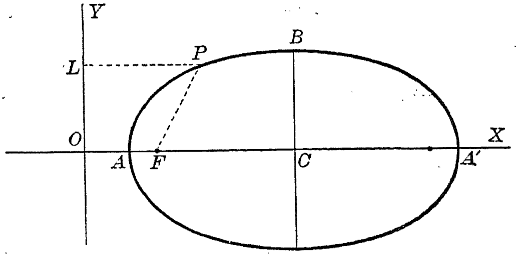


FIG. 35.

---

ellipse. *E.g.*, let $x = OM$; then the factors $\left(x - \frac{k}{1+e}\right)$ and $\left(\frac{k}{1-e} - x\right)$ are the two segments AM and MA' of the line AA', and geometrically their product equals the square of the ordinate MQ of the semicircle of which AA' is the diameter. If now the point P on MQ be so constructed that $MP = \sqrt{1-e^2} \cdot MQ$, then P is a point on the ellipse whose equation is (4) above.

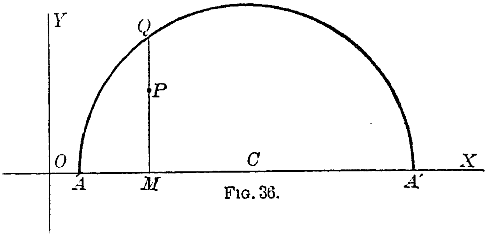


FIG. 36.

Similarly, any number of points on the curve can be constructed. This method shows also that the ordinates of an ellipse are less than, but in a constant ratio to, the corresponding ordinates of the circle of which the diameter is the line joining the vertices of the ellipse. See also Note B, Appendix.

increases from 0 to $\frac{ek}{\sqrt{1-e^2}}$ $\left(\text{which value it reaches when } x=\frac{k}{1-e^2}\right)$ and then decreases again to 0. The form of the curve is therefore as shown in Fig. 35, where $OF=k$, $OA=\frac{k}{1+e}$, $OC=\frac{k}{1-e^2}$, $OA'=\frac{k}{1-e}$, and $CB=\frac{ek}{\sqrt{1-e^2}}$.

(3) *The hyperbola*, $e>1$. Equation (1) may also be written in the form

$$y^2=(e^2-1)\left(x-\frac{k}{1+e}\right)\left(x-\frac{k}{1-e}\right), \quad . \quad . \quad . \quad (5)$$

which, when $e>1$, shows that y is imaginary for all values of x between $x=\frac{k}{1+e}$ and $x=\frac{k}{1-e}$, and that y is real for all other values of x. Equation (5) also shows that, as x increases from $\frac{k}{1+e}$ to ∞, y changes from 0 to $\pm\infty$, and that, as x decreases from $\frac{k}{1-e}$ to $-\infty$, y changes from 0 to $\pm\infty$. The form of the curve is therefore as shown in Fig. 37, where $OA=\frac{k}{1+e}$ and $OA'=\frac{k}{1-e}=-\frac{k}{e-1}$.

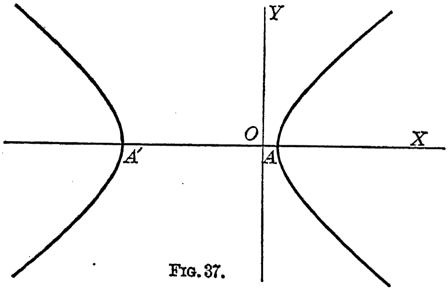


FIG. 37.

Although these three curves differ so widely in form, they are really very closely related as will be further shown in Chap. XII, and in Note D of the Appendix.

49. The use of curves in applied mathematics.* In Chapter III it was shown that whenever the relation between two variables, whose values depend upon each other, can be definitely stated, *i.e.*, when the variables can be connected by an equation, then the geometric or graphic representation of this relation is given by means of a curve. Such a curve often gives at a glance, information which would otherwise require considerable computation to secure; and in many cases it brings out facts of peculiar interest and importance which might otherwise escape notice.

The use of graphic methods in the study of physics and engineering, as well as in statistics and many other branches of investigation, is already extensive and is rapidly increasing. Under the name "graphic methods" there are included, however, not only such examples as those already given, where the equation connecting the variables is known, but also those where no such equation can be found; in these latter cases the curves constitute almost the only practical way of studying the relations involved.

As a simple example of this kind, suppose the temperature of a patient to be accurately observed at intervals of one hour; if the numbers representing the hours, *i.e.*, 1, 2, 3, ⋯ are taken as abscissas, and the *corresponding* numerical values of the temperatures be taken as ordinates, then a smooth curve drawn through the points so determined will express graphically the variation of the temperature of this patient with the time. This curve will also show to the physician what was the greatest and least temperature during the interval of the observations, as well as the time when each of these was attained. In this problem the curve gives no *new* information, but it presents in a much more concise and forcible form the information given by the tabulated numbers.

---

* For most of the suggestions in this article, and in the examples that follow it, the authors are indebted to Mr. J. S. Shearer of the Department of Physics of Cornell University

Again, if the distances passed over by a train in successive minutes during the run between two stations are taken as ordinates, and the corresponding number of minutes since starting, as abscissas, a smooth curve drawn through the points so determined will show at a glance, to an experienced eye, where and when additional steam was turned into the cylinders, brakes applied, heavy grades encountered, etc., etc.

In all such cases the coördinates of the points are taken to represent the numerical values of related quantities, such as time, length, weight, velocity, current, temperature, etc., and the curve through the points so determined usually gives, to an experienced person, all the information concerning the relations involved that is of practical importance. It is in the study of such curves that much of the value of training in analytic geometry becomes apparent to the physicist and the engineer. The student should early learn to translate physical laws into graphic forms, and he should give careful attention to the interpretation of all changes of form, intercepts, intersections, etc., of such curves.

EXERCISES

1. In simple interest if $p \equiv$ principal, $t \equiv$ time, $r \equiv$ rate, and $a \equiv$ amount, then $a = p\ (1 + rt)$. If now particular numerical values are given to p and r, and if the values of the variable a be taken as ordinates, and the corresponding values of t as abscissas, then the locus of this equation may be drawn. Draw this locus. What line in the figure represents the principal? What feature of the curve depends upon the rate per cent? Interpret the intercepts on the axes.

2. Give to p and r in exercise 1 different values and, with the same axes, draw the corresponding locus. How do these loci differ? What does their point of intersection mean?

3. With the same axes as before draw the curve for which *interest* and *time* are the coördinates; how is it related to the curves of exercises 1 and 2?

4. Draw and discuss the curve showing the relation between *amount*, *principal*, *rate*, and *time* in the case of compound interest.

(α) When interest is compounded annually.
(β) When interest is compounded quarterly.
(γ) When interest is compounded instantaneously.

5. A wage earner has already been working 10 days at $1.50 per day, and continues to do so 20 days longer, after which he is idle during 8 days; he then works 14 days more at the same wages, after which his employer raises his wages to $2.50 per day for the next 20 days: using the amounts earned as ordinates, and the time (in days) as abscissas, draw carefully the broken line which states the above facts.

What modification of the drawing would be necessary to show that the wage earner forfeited 50 cents per day during his idleness?

6. The following table shows the production of steel in Great Britain and the United States from 1878 to 1891.*

	U.S.	G.B.		U.S.	G.B.
1878 . .	7.3 (100,000 long tons)	10.6 (100,000 long tons)	1885 . .	17.1	19.7
1879 . .	9.3	10.9	1886 . .	25.6	23.4
1880 . .	12.5	13.7	1887 . .	33.4	31.5
1881 . .	15.9	18.6	1888 . .	29.0	34.0
1882 . .	17.4	21.9	1889 . .	33.8	36.7
1883 . .	16.7	20.9	1890 . .	42.8	36.8
1884 . .	15.5	18.5	1891 . .	39.0	32.5

Using time (in years) as abscissas, and quantity of steel produced (100,000 tons per unit) as ordinates, the separate points represented by the table have been plotted (Fig. 38) and then joined by straight lines, dotted for Great Britain and full for the United States.*

---

* Taken by permission from Lambert's Analytic Geometry.

Interpret fully the figure.

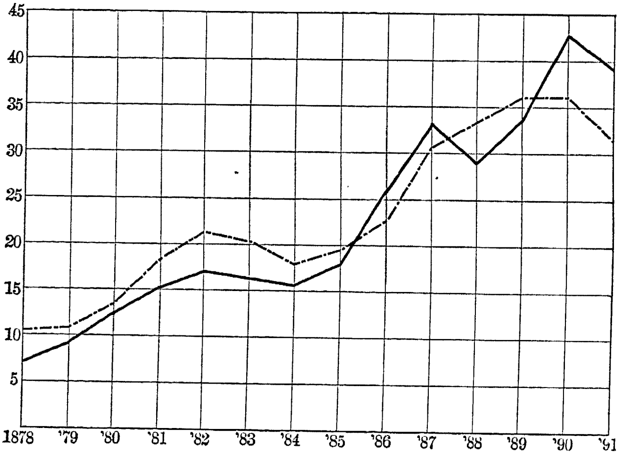


FIG. 38.

7. Exhibit graphically the information contained in the following table on the expense of moving freight per "ton-mile" on N. Y. C. & H. R. R. R. from 1866 to 1893.

1866	2.16¢	1873	1.03¢	1880	.54¢	1887	.56¢
1867	1.95	1874	.98	1881	.56	1888	.59
1868	1.80	1875	.90	1882	.60	1889	.57
1869	1.40	1876	.71	1883	.68	1890	.54
1870	1.15	1877	.70	1884	.62	1891	.57
1871	1.01	1878	.60	1885	.54	1892	.54
1872	1.13	1879	.55	1886	.53	1893	.54

8. The following table gives the population of the countries named between 1810 and 1896: †

---

* In the figure the linear unit on the x-axis is 5 times as long as the linear unit on the y-axis. It will, however, be noticed that the essential feature of a system of coördinates, the "one-to-one correspondence" of the symbol (x, y) and the points of a plane, is not disturbed by using different scales for ordinates and abscissas.

† The authors are indebted to Professor W. F. Willcox of Cornell University for these data, which are compiled from the *Statesman's Year Book* for 1897, and from *Statistik des Deutschen Reichs*, Bd. 44, 1892.

British Isles		Lands now included in the German Empire	
Year	Population	Year	Population
1801	15,896,000	1816	24,831,000
1811	17,908,000	1837	31,540,000
1821	20,894,000	1847	34,753,000
1831	24,029,000	1856	36,130,000
1841	26,709,000	1865	39,399,000
1851	27,369,000	1872	41,028,000
1861	28,927,000	1876	42,775,000
1871	31,485,000	1885	46,856,000
1881	34,885,000	1895	52,280,000
1891	37,733,000		

France		Ireland		United States	
Year	Population	Year	Population	Year	Population
1821	30,462,000	1811	5,938,000	1810	7,240,000
1841	34,230,000	1821	6,802,000	1820	9,634,000
1861	37,386,000	1831	7,767,000	1830	12,866,000
1866	38,067,000	1841	8,175,000	1840	17,069,000
1872	36,103,000	1851	6,552,000	1850	23,192,000
1876	36,906,000	1861	5,799,000	1860	31,443,000
1881	37,672,000	1871	5,412,000	1870	38,558,000
1886	38,219,000	1881	5,175,000	1880	50,156,000
1891	38,343,000	1891	4,705,000	1890	62,622,000
1896	38,518,000				

Employing the number of years as abscissas, and the population (500,000 per unit, — numbers at left of figure represent millions) as ordinates, the separate points represented by the above table have been plotted (Fig. 39) and then joined by straight lines. The figure gives all the information contained in the tabulated results, besides showing at a glance the relative population of the different countries at any given time. The student may account historically for the abrupt fall in the line representing the population of France; and for the gradual downward tendency in the line representing the population of Ireland.

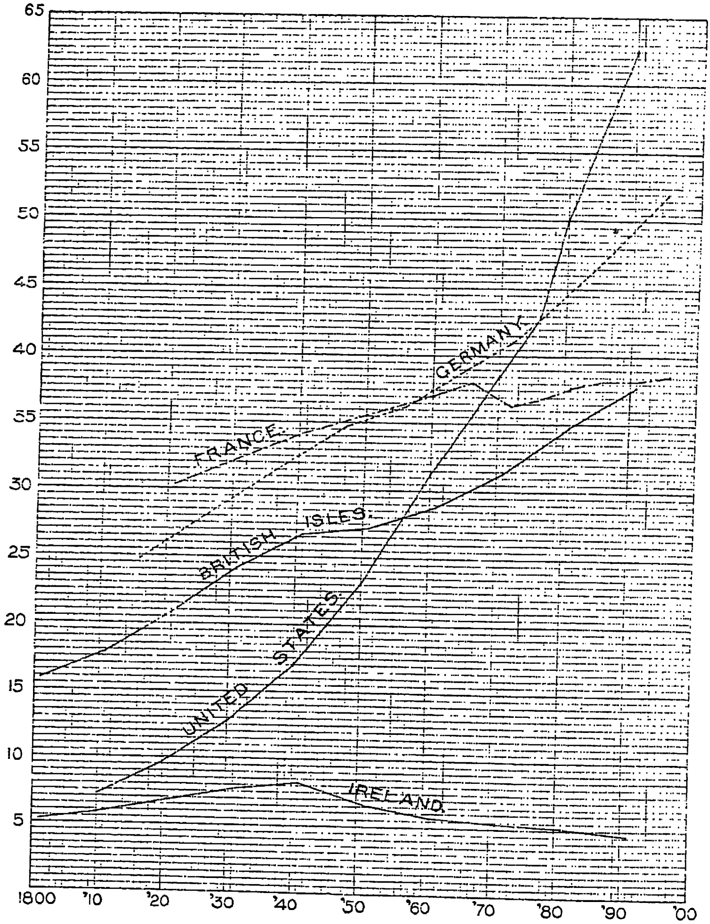

65
60
55
50
45
40
35
30
25
20
15
10
5
1800
'10
'20
'30
'40
'50
'60
'70
'80
'90
'00
GERMANY
FRANCE
BRITISH ISLES
UNITED STATES
IRELAND

Fig. 39.

EXAMPLES ON CHAPTER IV

1. Find the equations of the sides of the triangle whose vertices are the points $(2, 3)$, $(4, {}^{-}5)$, $(3, {}^{-}6)$ (cf. Art. 43). Test the resulting equations by substitution of the given coördinates.

2. Find the equations of the sides of the square whose vertices are $(0, {}^{-}1)$, $(2, 1)$, $(0, 3)$, $({}^{-}2, 1)$. Compare the equations of the parallel sides; of perpendicular sides.

3. Find the coördinates of the center of the square in Ex. 2. Then find the radius of the circumscribed circle, and (Art. 47) the equation of that circle. Test the result by finding the coördinates of the points of intersection of one of the sides with circle (Art. 39).

4. Find the equation of the path traced by a point which is always equidistant from the points

(α) $(2, 0)$ and $(0, {}^{-}2)$; (β) $(3, 2)$ and $(6, 6)$;

(γ) $(a + b, a - b)$ and $(a - b, a + b)$.

5. A point moves so that its ordinate always exceeds $\frac{4}{3}$ of its abscissa by 6. Find the equation of its locus, and trace the curve.

6. A point moves so that the square of its ordinate is always 4 times its abscissa. Find the equation of its locus and trace the curve.

7. Find the equation of the locus of a point which moves so that the sum of its distances from the points $(1, 3)$ and $(4, 2)$ is always 5. Trace and discuss the curve.

8. Find the equation of the locus of the point in example 7, if the difference of its distances from the fixed points is always 2.

9. Express by a single equation the fact that a point moves so that its distance from the x-axis is always numerically 3 times its distance from the y-axis.

10. A point moves so that the square of its distance from the point $(a, 0)$ is 4 times its ordinate. Find the equation of its locus, and trace the curve.

11. A point moves so that its distance from the x-axis is $\frac{1}{2}$ of its distance from the origin. Find the equation of its locus, and trace the curve.

12. A point moves so that the difference of the squares of its distances from the points $(1, 3)$ and $(4, 2)$ is 5. Find the equation of its locus and trace the curve.

13. Solve example 12 if the word "sum" is substituted for "difference."

14. Let $A \equiv (a, 0)$, $B \equiv (b, 0)$, and $A' \equiv (-a, 0)$ be three fixed points; find the equation of the locus of the point $P \equiv (x, y)$ which moves so that $\overline{PB}^2 + \overline{PA}^2 = 2\,\overline{PA'}^2$.

15. A point moves so that $\frac{1}{4}$ of its abscissa exceeds $\frac{1}{5}$ of its ordinate by 1. Find the equation of its locus and trace the curve.

16. Find the equation of the locus of a point that is always equidistant from the points $(-3, 4)$ and $(5, 3)$; from the points $(-3, 4)$ and $(2, 0)$. By means of these two equations find the coördinates of the point that is equidistant from the three given points.

17. Let $A \equiv (-1, 3)$, $B \equiv (-3, -3)$, $C \equiv (1, 2)$, $D \equiv (2, 2)$ be four fixed points, and let $P \equiv (x, y)$ be a point that moves subject to the condition that the triangles PAB and PCD are always equal in area; find the equation of the locus of P.

18. If the area of a triangle is 25 and two of its vertices are $(5, -6)$ and $(-3, 4)$, find the equation of the locus of the third vertex.

19. A point moves so that its distance from the pole is numerically equal to the tangent of the angle which the straight line joining it to the origin makes with the initial line. Find the polar equation of its locus and plot the figure.

CHAPTER V

THE STRAIGHT LINE. EQUATION OF FIRST DEGREE
$Ax + By + C = 0$

50. In Chapter III it was shown that to every equation between two variables there corresponds a definite geometric locus, and in Chapter IV it was shown that if the geometric locus be given, its equation may be found. It still remains to exhibit in greater detail some of the more elementary loci and their equations, and to apply analytic methods to the study of the properties of these curves. Since the straight line is a simple locus, and one whose properties are already well understood by the student, its equation will be examined first.

In studying the straight line, as well as the circle and other second degree curves, to be taken up in later chapters, it will be found best first to obtain the simplest equation which represents the locus, and to study the properties of the curve from that simple or *standard* equation. Then it remains to find methods for reducing to this standard form any other equation that represents the same locus.

51. Equation of straight line through two given points. A numerical example of the equation of the line through two fixed points has already been given in Art. 43; in the present article the equation of a straight line through *any* two given points will be derived; the method, however, will be precisely the same as that already employed in the numerical example.

Let the two given fixed points be $P_1 \equiv (x_1, y_1)$ and $P_2 \equiv (x_2, y_2)$, and let $P \equiv (x, y)$ be *any* other point on the line through P_1 and P_2. Draw the ordinates M_1P_1, M_2P_2, and MP; also through P_1 draw P_1R_2 parallel to the x-axis, and meeting MP in R and M_2P_2 in R_2. Then the triangles P_1RP and $P_1R_2P_2$ are similar;

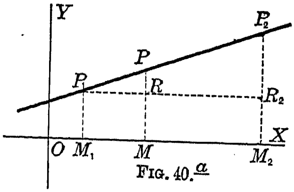


FIG. 40.a

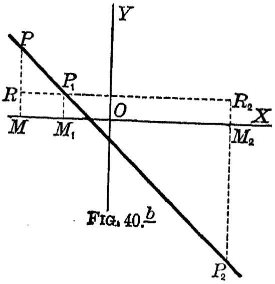


FIG. 40.b

$$\therefore \quad \frac{RP}{R_2P_2} = \frac{P_1R}{P_1R_2}, \quad i.e., \quad \frac{MP - M_1P_1}{M_2P_2 - M_1P_1} = \frac{OM - OM_1}{OM_2 - OM_1}.$$

Substituting in this last equation the coördinates of P_1, P_2, and P, it becomes

$$\boldsymbol{\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}}; \quad . \quad . \quad . \quad [9]$$

and since $P \equiv (x, y)$ is *any* point on the line through P_1 and P_2, therefore equation [9] is satisfied by the coördinates of *every* point on this line. That equation [9] is not satisfied by the coördinates of any point except such as are on the line P_1P_2 may be proved as was done in Art. 43.

Equation [9] then fulfills both requirements of the definition in (1) of Art. 35, and is therefore the equation of the straight line through the two points (x_1, y_1) and (x_2, y_2). This equation will be frequently needed and will be referred to as a *standard form;* it should be committed to memory.*

---

* Throughout this book the more important formulas are printed in bold-faced type; they should be committed to memory by the learner.

52. Equation of straight line in terms of the intercepts which it makes on the coördinate axes. If the two given

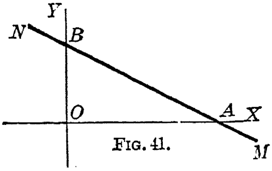


FIG. 41.

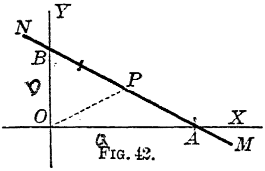


FIG. 42.

points in Art. 51 are those in which the line cuts the axes of coördinates, *i.e.*, $A \equiv (a, 0)$ and $B \equiv (0, b)$ (Fig. 41), then equation [9] becomes

$$\frac{y-0}{b-0} = \frac{x-a}{0-a};$$

that is,

$$\frac{x}{a} + \frac{y}{b} = 1, \qquad . \quad . \quad . \qquad [10]$$

where a and b are the intercepts which the line cuts from the axes.

This is another standard form of the equation of the straight line; it is known as the **symmetrical** or the **intercept** form.

Equation [10] may also be derived independently of equation [9] thus: let the line MN (Fig. 42), whose equation is to be found, cut the axes at the points $A \equiv (a, 0)$ and $B \equiv (0, b)$, and let $P \equiv (x, y)$ be *any* other point on this line. Connect O and P; then

$$\text{area } OPB + \text{area } OAP = \text{area } OAB;$$

that is,

$$\tfrac{1}{2}\,bx + \tfrac{1}{2}\,ay = \tfrac{1}{2}\,ab,$$

and, dividing by $\frac{1}{2}\,ab$, this equation becomes $\frac{x}{a} + \frac{y}{b} = 1$, as above.

EXERCISES

1. Show that equation [10] is not satisfied by the coördinates of any point except those lying on MN.

2. Write down the equations of the lines through the following pairs of points:

(α) (3, 4) and (5, 2); (γ) (-6, 1) and (-2, -5);

(β) (3, 4) and (5, -2); (δ) (-15, -3) and $\left(\frac{8}{3}, \frac{-7}{9}\right)$.

3. Write the equations of the lines which make the following intercepts on the x- and y-axes respectively.

(α) 4 and 7; (β) -3 and 5; (γ) $\frac{4}{3}$ and $-\frac{1}{2}$; (δ) $-\frac{a}{2}$ and $3a$.

4. What do equations [9] and [10] become if one of the given points is the origin?

5. By drawing, in Fig. 42, a perpendicular PM from P to the x-axis, derive equation [10] from the similar triangles MAP and OAB.

6. Is equation [10] true if P is on MN but not between A and B?

7. Are equations [9] and [10] true if the coördinate axes are not at right angles to each other?

8. Is the point (3, $4\frac{1}{2}$) on the line through the points (2, 3) and (5, 7)? On which side of this line is it? Which is the negative side of this line?

9. What intercepts does the line through the points (1, -6) and (-3, 5) make on the axes?

10. The vertices of a triangle are: (4, -5), (2, 3), and (3, -6). Find the equations of the sides; also of the three medians; then find the coördinates of the point of intersection of two of these medians, and show that these coördinates satisfy the equation of the other median. What proposition of plane geometry is thus proved?

11. Find the tangent of the angle (the "slope," cf. Art. 27) which the line in exercise 9 makes with the x-axis.

12. Draw the line whose equation is $\frac{x}{2} + \frac{y}{3} = 1$, and then find the equations of the two lines which pass through the origin and trisect that portion of this line which lies in the first quadrant.

53. Equation of straight line through a given point and in a given direction (cf. Art. 44). Let $P_1 \equiv (x_1, y_1)$ be the given point, and let the direction of the line be given by the angle $XAP = \theta$ which the line makes with the x-axis; also let $P \equiv (x, y)$ be *any* point on the given line and denote the slope, *i.e.*, $\tan\theta$, by m. Draw the ordinates

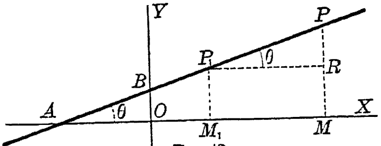


FIG. 43.

M_1P_1 and MP, and through P_1 draw P_1R parallel to the x-axis and meeting the ordinate MP in R.

Then, in triangle RP_1P, the angle $RP_1P = \theta$;

hence $$m = \tan\theta = \frac{RP}{P_1R} = \frac{y - y_1}{x - x_1}.$$

$$\text{[Since } RP = y - y_1 \text{ and } P_1R = x - x_1\text{];}$$

that is, $$y - y_1 = m(x - x_1), \quad . \quad . \quad . \quad [11]$$

which is the desired equation.

COR. If the given point be $B \equiv (0, b)$, *i.e.*, the point in which the line meets the y-axis, then equation [11] becomes

$$y = mx + b. \quad . \quad . \quad . \quad [12]$$

Equation [12] is usually spoken of as the **slope** form of the equation of the straight line.

EXERCISES

1. What do the constants m and b in equation [12] mean? Draw the line for which $m = 4$ and $b = 3$; also that for which $m = -1$ and $b = -\frac{3}{2}$.

2. What is the effect on the line [12] of a change in b while m remains the same? What if m be changed and b left unchanged?

3. Describe the effect on the line [11] of changing m while x_1 and y_1 remain the same; also the effect resulting from a change in x_1 while m and y_1 remain the same.

4. Write the equation of a line through the point $(^-3, 7)$, and making with the x-axis an angle of 30°; of $^-30°$; of $\left(\frac{2\pi}{3}\right)^{(r)}$; of $\left(\frac{7\pi}{6}\right)^{(r)}$.

5. Write the equations of the following lines:

(α) slope 3, y-intercept 8; (β) slope $\frac{1}{2}$, y-intercept $^-3$;

(γ) slope $^-2$, y-intercept $\frac{3}{4}$.

6. A line has the slope 6; what is its y-intercept if it passes through the point (7, 1)?

7. What must be the slope of a line whose y-intercept is $^-3$, in order that it may pass through the point ($^-5$, 5)?

8. Is the point $(1, \frac{}{2})$ on the line passing through the point ($^-2$, $^-14$), and making an angle $\tan^{-1}\frac{11}{2}$ with the x-axis?

9. How do the lines $y = 3x - 1$, $y = 3x + 7$, and $2y - 6x + 15 = 0$ differ from each other? What have they in common? Draw these lines.

10. What is common to the lines $y = 3x - 1$, $2y = 5x - 2$, and $7x - 3y = 3$?

11. What is the slope of [9]? of [10]?

12. Derive equation [12] independently of equation [11].

54. Equation of straight line in terms of the perpendicular from the origin upon it, and the angle which that perpendicular makes with the x-axis. Let HK be the line whose equation is sought, and let the perpendicular ($ON = p$) from O upon this line, and the angle (a) which this perpendicular makes with the x-axis, be given. Also let $P \equiv (x, y)$ be any point on HK; then by projection upon ON (Art. 17),

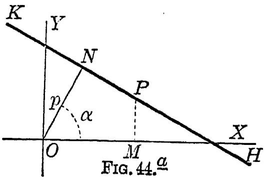


FIG. 44.a

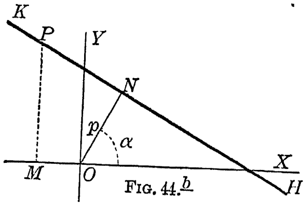


FIG. 44.b

$$OM \cos a + MP \sin a = ON,$$

i.e.,
$$x \cos \alpha + y \sin \alpha = p, \quad . \quad . \quad . \quad [13]$$

which is the required equation.

Equation [13] is known as the **normal** form of the equation of the straight line.

In the following pages p will always be regarded as positive, and a as positive and less than 360°.

55. Normal form of equation of straight line: second method. The student should bear in mind that to get the equation of a curve, he has merely to obtain an equation that is satisfied by the coördinates of every point on the curve, and not satisfied by the coördinates of any other point; and that it is wholly immaterial what particular geometric property he may employ in the accomplishment of this purpose. This fact is already illustrated in Art. 52, where equation [10] was obtained in two ways, while Ex. 5, p. 84, gives still a third method by which the same equation may be found. So also it is possible to derive equation [13] by other methods than that employed in Art. 54.*

E.g., in Fig. 41 draw a perpendicular from O to the line AB, let its length be denoted by p, and let α be the angle which it makes with the x-axis, then

$$a \cos \alpha = p, \text{ and } b \sin \alpha = p,$$

whence

$$a = \frac{p}{\cos \alpha}, \text{ and } b = \frac{p}{\sin \alpha}.$$

Substituting these values of a and b in equation [10], it becomes

$$\frac{x}{\dfrac{p}{\cos \alpha}} + \frac{y}{\dfrac{p}{\sin \alpha}} = 1, \; i.e., \; x \cos \alpha + y \sin \alpha = p,$$

which is the form already derived in Art. 54.

NOTE. In Art. 2, constants, variables, etc., were illustrated by means of a triangle. Now that the student has learned that the equation $\frac{x}{a} + \frac{y}{b} = 1$, for example, represents a straight line, *i.e.*, that this equation is satisfied by all those pairs of values of x and y which are the coördinates of points on this line, a somewhat better illustration can be given. Both x and y are variables, but are not independent; each is an implicit function of the other. For any particular line a and b are constants, but they may represent other constants in the equation of another line, *i.e.*, they are arbitrary constants, and are often called **parameters** of the line.

---

* See also Ex. 6 below.

EXERCISES

1. The perpendicular from the origin upon a certain line is 5; this perpendicular makes an angle of $\frac{\pi}{3}$ with the x-axis; what is the equation of the line?

2. If in equation [13] p is increased while α remains the same, what is the effect upon the line? If α be changed while p remains the same, what is the effect?

3. A certain line is 3 units distant from the origin, and makes an angle of 120° with the x-axis; what is its equation?

4. Given $\alpha = 30°$, what must be the length of p in order that the line HK (see Fig. 44 a) shall pass through the point (7, 2)?

5. A line passes through the point $(^-3, {}^-4)$, and a perpendicular upon it from the origin makes an angle of 45° with the x-axis. What is the equation of this line?

6. In Fig. 44 a draw through M a line parallel to HK, meeting ON in R; then draw through P a perpendicular to MR, meeting it in Q; by means of the figure so constructed derive equation [13] anew.

56. Summary. The results of Arts. 51–55 may be briefly summarized thus:

The position of a straight line is determined by: (1) two points through which it passes; (2) one point and the direction in which the line passes through this point. Under (1) there is the special case in which the two given points are one on the x-axis and the other on the y-axis. Under (2) there are two special cases: (α) when the given point is on an axis (the y-axis say), and (β) when the point is given by its distance and direction from the origin, while the line whose equation is sought is perpendicular to the line which connects the given point to the origin.

Corresponding to these two general and three special cases, there have been derived five standard forms of the equation of the straight line, viz.: equations [9], [10], [11], [12], and [13].

It may be remarked that equations [9] and [10] are independent of the angle between the coördinate axes, while [11],

[12], and [13] (m, α, and p retaining their present meanings) are true only when the axes are rectangular. It may also be pointed out that, from the nature of its derivation, equation [9] is inapplicable when the line is parallel to either axis; equation [10] is inapplicable when the line passes through the origin; and equations [11] and [12] are not applicable when the line is parallel to the y-axis.

57. Every equation of the first degree between two variables has for its locus a straight line. It will probably not have escaped the reader's notice that the five "standard" equations (equations [9] to [13]) of the straight line, which have been derived in Arts. 51 to 54, are each of the first degree. It will now be shown that every equation of the first degree between two variables has a straight line for its locus. The most general equation of this kind may be written in the form

$$Ax + By + C = 0, \quad . \quad . \quad . \quad (1)$$

where A, B, and C are constants, and neither A nor B is zero.*

Let $P_1 \equiv (x_1, y_1)$, $P_2 \equiv (x_2, y_2)$, and $P_3 \equiv (x_3, y_3)$ be *any* three points on the locus of equation (1). Draw the ordinates M_1P_1, M_2P_2, and M_3P_3; also draw HP_2 and KP_3 parallel to the x-axis.

Then, by Art. 35 (1),

$$Ax_1 + By_1 + C = 0 \ldots (2)$$

$$Ax_2 + By_2 + C = 0 \ldots (3)$$

$$Ax_3 + By_3 + C = 0 \ldots (4)$$

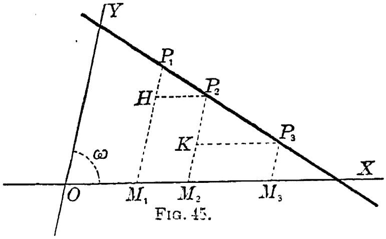


FIG. 45.

---

* If either A or B, say A, is zero, then the equation may be written in the form: $y = -\frac{C}{B}$, which is the equation of a straight line parallel to the x-axis, and at the distance $-\frac{C}{B}$ from it [cf. Art. 33, (2)].

By subtracting eq. (3) from eq. (2), and also eq. (4) from eq. (3), the two equations

$$A(x_1 - x_2) + B(y_1 - y_2) = 0,$$

and
$$A(x_2 - x_3) + B(y_2 - y_3) = 0,$$

are obtained. These give

$$\frac{y_1 - y_2}{x_1 - x_2} = -\frac{A}{B}, \text{ and } \frac{y_2 - y_3}{x_2 - x_3} = -\frac{A}{B}; \quad . \quad . \quad . \quad (5)$$

hence,
$$\frac{y_1 - y_2}{x_1 - x_2} = \frac{y_2 - y_3}{x_2 - x_3}. \quad . \quad . \quad . \quad (6)$$

But
$$y_1 - y_2 = HP_1, \qquad x_1 - x_2 = -M_1M_2 = -HP_2,$$
$$y_2 - y_3 = KP_2, \text{ and } x_2 - x_3 = -M_2M_3 = -KP_3;$$

hence, from eq. (6),
$$\frac{HP_1}{HP_2} = \frac{KP_2}{KP_3}.$$

Also, by construction,

$$\angle P_2HP_1 = \angle P_3KP_2;$$

hence, triangle HP_2P_1 is similar to triangle KP_3P_2,

and
$$\angle P_1P_2H = \angle P_2P_3K;$$

$$\therefore \angle P_1P_2H + \angle HP_2K + \angle KP_2P_3$$
$$= \angle P_2P_3K + \angle P_3KP_2 + \angle KP_2P_3 = 2 \text{ rt. } \angle\text{s};$$

i.e., P_2 lies on the straight line joining P_1 and P_3. But, since P_2 is *any* point on the locus of $Ax + By + C = 0$, hence *all* points of this locus lie on the same straight line P_1P_3, which, therefore, constitutes the locus of $Ax + By + C = 0$.

Since this demonstration does not depend upon the angle ω, therefore it applies whether the axes are oblique or rectangular; hence the theorem: *every equation of the first degree between two variables, when interpreted in Cartesian coördinates, represents a straight line.*[*]

---

[*] This conclusion may also be drawn thus: clear equation (6) of fractions, transpose all the terms to the first member, and multiply by $\frac{1}{2}\sin\omega$;

Because of this fact, such an equation is often spoken of as a **linear** equation.

NOTE. In the equation $Ax + By + C = 0$, there are apparently three constants; in reality, there are but two independent constants, viz. the ratios of the coefficients (cf. Art. 38). This corresponds to the fact that a straight line is determined geometrically by two conditions.

58. Reduction of the general equation $Ax + By + C = 0$ to the standard forms. Determination of a, b, m, p, and α in terms of A, B, and C.*

(1) *Reduction to the standard form* $\frac{x}{a}+\frac{y}{b}=1$ *(symmetric or intercept form).*

That the equation

$$Ax + By + C = 0 \quad . \quad . \quad . \quad (1)$$

represents *some* straight line has just been shown (Art. 57); again, since multiplication by a constant, and transposition, do not change the locus (Art. 38), therefore

$$\frac{x}{-\frac{C}{A}}+\frac{y}{-\frac{C}{B}}=1\dagger \quad . \quad . \quad . \quad (2)$$

represents the same line. But equation (2) is in the required form (Art. 52), and its intercepts are:

$$a = -\frac{C}{A}, \text{ and } b = -\frac{C}{B}.$$

(2) *Reduction to the standard form* $y = mx + b$ *(slope form).*

---

the resulting equation asserts [see Art. 29, (1)] that the area of the triangle formed by the points P_1, P_2, and P_3, is zero; *i.e.*, these three points lie on a straight line; but they are *any* three points on the locus of $Ax + By + C = 0$, hence that locus is a straight line.

* These reductions constitute a second proof of the theorem of Art. 57.

† If $C = 0$, the line represented by (1) goes through the origin, and the symmetric form of the equation is inapplicable (Art. 56); but, in that case, the above reduction also fails, since it is not permissible to divide the members of an equation by zero.

The equation $Ax + By + C = 0$ has the same locus as has the equation

$$y = \left(-\frac{A}{B}\right)x + \left(-\frac{C}{B}\right)^* \quad . \quad . \quad . \quad (3)$$

(see Art. 38); but this is the equation (Art. 53) of a line drawn through the point $\left(0, -\frac{C}{B}\right)$, and making with the x-axis the angle $\theta = \tan^{-1}\left(-\frac{A}{B}\right)$; hence equation (3) is in the required form, and

$$m = -\frac{A}{B}, \text{ and } b = -\frac{C}{B}.$$

(3) *Reduction to the standard form* $x\cos\alpha + y\sin\alpha = p$ (*normal form*).

If equation (1) and

$$x\cos\alpha + y\sin\alpha = p \quad . \quad . \quad . \quad (4)$$

represent the same line, then they differ merely by some constant multiplier, say k (cf. Art. 38). Then

$$kAx + kBy + kC \equiv x\cos\alpha + y\sin\alpha - p = 0;$$

$$\therefore\ kA = \cos\alpha,\ kB = \sin\alpha, \text{ and } kC = -p;$$

$$\therefore\ k^2A^2 + k^2B^2 = \cos^2\alpha + \sin^2\alpha = 1;$$

whence $$k = \frac{1}{\sqrt{A^2+B^2}};$$

hence $$\cos\alpha = \frac{A}{\sqrt{A^2+B^2}},\ \sin\alpha = \frac{B}{\sqrt{A^2+B^2}},$$

and $$p = -\frac{C}{\sqrt{A^2+B^2}},$$

---

[*] If $B = 0$, the line represented by equation (1) is parallel to the y-axis, and the slope form of the equation is inapplicable (Art. 56); but, in that case, the above reduction also fails.

wherein the algebraic sign of $\sqrt{A^2+B^2}$ is to be chosen so as to make $\frac{-C}{\sqrt{A^2+B^2}}$ positive, since p is to be always positive (Art. 54); *i.e.*, the sign of $\sqrt{A^2+B^2}$ is to be opposite to that of the number represented by C.

Hence, to reduce equation (1) to the normal form, *i.e.*, to the form of equation (4), it is only necessary to divide equation (1) by $\sqrt{A^2+B^2}$, with the sign properly chosen, and transpose the constant term to the second member. This gives

$$\frac{A}{\sqrt{A^2+B^2}}x+\frac{B}{\sqrt{A^2+B^2}}y=\frac{-C}{\sqrt{A^2+B^2}}.$$

(4) *Another method for reduction to the normal form.*

If the equation $Ax+By+C=0$ and $x\cos a+y\sin a=p$ represent the same line, then they must have the same y-intercept and the same slope, *i.e.*,

$$-\frac{C}{B}=\frac{p}{\sin a}, \quad . \quad . \quad . \quad (5)$$

and

$$-\frac{A}{B}=-\frac{\cos a}{\sin a}. \quad . \quad . \quad . \quad (6)$$

Squaring eq. (6), and adding 1 to each member, gives

$$\frac{A^2+B^2}{B^2}=\frac{\cos^2 a+\sin^2 a}{\sin^2 a}$$

$$=\frac{1}{\sin^2 a};$$

$$\therefore \ \sin a=\frac{B}{\sqrt{A^2+B^2}};$$

whence

$$\cos a=\frac{A}{\sqrt{A^2+B^2}} \text{ and } p=\frac{-C}{\sqrt{A^2+B^2}}.$$

as before. These, then, are the values of p, $\sin\alpha$, and $\cos\alpha$, which are to be substituted in $x\cos\alpha + y\sin\alpha = p$.

Hence $$\frac{A}{\sqrt{A^2+B^2}}x + \frac{B}{\sqrt{A^2+B^2}}y = -\frac{C}{\sqrt{A^2+B^2}},$$

is an equation representing the same locus as $Ax + By + C = 0$, and having the normal form.

59. To trace the locus of an equation of the first degree. In Art. 57 it was proved that the locus of an equation of the first degree in two variables is a straight line; but a straight line is fully determined by any two points on it; hence, to trace the locus of a first degree equation it is only necessary to determine two of its points, and then to draw the indefinite straight line through them. The two points most easily determined, and plotted, are those in which the locus cuts the axes; they are therefore the most advantageous points to employ. If the line is parallel to an axis, then only one point is needed.

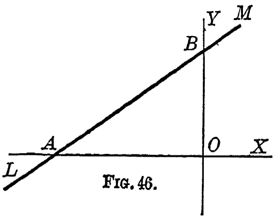


FIG. 46.

E.g., to trace the locus of the equation

$$2x - 3y + 12 = 0:$$

the ordinate of the point in which this line crosses the x-axis is 0; let its abscissa be x_1, then $(x_1, 0)$ must satisfy the equation $2x - 3y + 12 = 0$;

hence $$2x_1 - 3\cdot 0 + 12 = 0,$$

whence $$x_1 = -6,$$

i.e., the line crosses the x-axis at the point $(^-6, 0)$. In like manner it is shown that it crosses the y-axis at the point $(0, 4)$. Therefore LM is the locus of $2x - 3y + 12 = 0$.

60. Special cases of the equation of the straight line $Ax + By + C = 0$. This equation, written in the intercept form [Art. 58 (1)] becomes

$$\frac{x}{-\frac{C}{A}} + \frac{y}{-\frac{C}{B}} = 1, \quad . \quad . \quad . \quad (1)$$

If in equation (1), A is made smaller and smaller in comparison with C, then the x-intercept $\left(-\frac{C}{A}\right)$ becomes larger and larger; if $A \doteq 0$ in comparison with C, the x-intercept grows infinitely large, the line (1) becomes parallel to the x-axis, and its equation becomes

$$\frac{x}{\infty} + \frac{y}{-\frac{C}{B}} = 1; \ i.e., \ y = -\frac{C}{B},$$

which agrees with the foot-note of Art. 57.

Similarly, if $B \doteq 0$ in comparison with C, the line (1) becomes parallel to the y-axis, and its equation becomes

$$x = -\frac{C}{A}.$$

If both A and B approach zero simultaneously in comparison with C, then both the intercepts become indefinitely large, and the line (1) recedes farther and farther from the origin.

In accordance with what has just been said, a line that is wholly at infinity might have its equation written in the form

$$0 \cdot x + 0 \cdot y + C = 0, \quad . \quad . \quad . \quad (2)$$

or, as it is sometimes written,

$$C = 0; \quad . \quad . \quad . \quad (3)$$

but equations (2) and (3) are merely abbreviations for the statement: "As both A and B approach zero in comparison with C, the line moves farther and farther from the origin."

EXERCISES

1. Reduce the following equations to the intercept (symmetric) form, and draw the lines which they represent:

(α) $3x - 2y + 12 = 0$; (β) $3x - 2y + 1 = 5x + 3$;

(γ) $2y = 15 - y + 5x$; (δ) $\dfrac{x - 2y + 1}{3 + 7y} = 9$.

2. Reduce to the slope form, and then trace the loci:

(α) $7x - 5y + 6(y - 3x) = -10x + 4$; (β) $3x + 2y + 6 = 0$;

(γ) $3x + 5 = 3 - y$.

Which is the positive side of the line (β)? (cf. foot-note, Art. 43.)

3. Reduce to the normal form, and then trace the loci:

(α) $3x + 4y = 15$; (β) $3x - 4y + 15 = 0$;

(γ) $x - 3y = 5 + 6x$; (δ) $\frac{2}{3}x = y - 5$.

4. Show that the lines $3x + 5 = y$ and $6x - 2y = 81$ are parallel.

5. What is the slope of the line between the two points $(3, {}^{-}1)$ and $(2, 2)$? What is its distance from the origin? Which is its negative side?

6. A line passes through the point (5, 6) and has its intercepts on the axes equal and both positive. Find its equation and its distance from the origin.

7. A straight line passes through the point $(1, {}^{-}2)$ and is such that the portion of it between the axes is bisected by that point. What is the slope of the line?

8. What are the intercepts which the line through the points $({}^{-}1, 3)$ and (6, 7) makes on the axes? Through the points $(a, 2a)$ and $(b, 2b)$?

9. What system of lines obtained by varying the parameter b is represented by the equation $y = 6x + b$?

10. What system of lines obtained by varying the parameter m is represented by the equation $y = mx + 6$?

11. What family (system) of lines obtained by varying the parameter α is represented by the equation $x \cos\alpha + y \sin\alpha = 5$? To what curve is each line of the family tangent?

12. Find $\cos\alpha$ and $\sin\alpha$ for the lines

(α) $y = mx + b$, (β) $\dfrac{x}{a} + \dfrac{y}{b} = 1$,

(γ) $\dfrac{3}{x} = \dfrac{2}{y}$, (δ) $7x - 5y + 1 = 0$.

13. Find by means of cos α and sin α what quadrant is crossed by each of the lines:

(α) $3x+2=2y$; (β) $5x+3y+15=0$; (γ) $x-\sqrt{3}y-10=0$.

14. What must be the slope of the line $4x-ky=17$ in order that it shall pass through the point (1, 3)? Can k be determined so that the line will pass through the origin?

15. Determine the values of A, B, C in order that the line

$$Ax+By+C=0$$

shall pass through the points (3, 0) and $(0, {}^-12)$. [Art. 57, Note.]

16. Derive equation [9] by supposing (x_1, y_1) and (x_2, y_2) to be two points on the line $y=mx+b$; and thence finding values for m and b.

17. Find the slopes of the lines $2y-3x=7$ and $3y+2x-11=0$; and thence show that these lines are perpendicular to each other.

18. Find cos α for each of the lines $7x+y-9=0$ and $x-7y+2=0$, and then show that the two lines are perpendicular to each other.

19. Show by means of: (1) the slopes; (2) the angles; that the lines

$$2y-3x=7, \qquad 2y-3x+5=0, \qquad 10y-15x+c=0$$

are all parallel.

20. Reduce the equation $Ax+By+C=0$ to the normal form, *i.e.*, to the form $x\cos\alpha+y\sin\alpha=p$. Suggestion: the two equations as representing the same line, make the same intercepts on the axes.

61. To find the angle made by one straight line with another. Let the equations of the lines be

$$y=m_1x+b_1, \quad . \quad . \quad . \quad (1)$$

and $$y=m_2x+b_2, \quad . \quad . \quad . \quad (2)$$

where $m_1=\tan\theta_1$, $m_2=\tan\theta_2$, and θ_1, θ_2 are the angles which these lines make, respectively, with the x-axis. It is required to find the angle ϕ, measured from line (2) to line (1).

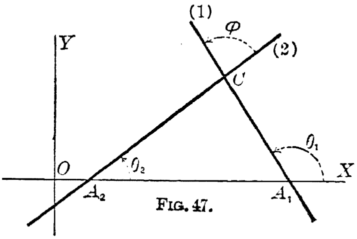


FIG. 47.

Since $$\phi = \theta_1 - \theta_2,$$

$$\therefore \qquad \tan\phi = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \cdot \tan\theta_2}, \qquad \text{(Art. 16)}$$

i.e., $$\tan\phi = \frac{m_1 - m_2}{1 + m_1 m_2}. \qquad [14]$$

If the angle were measured from line (1) to line (2) it would be the negative, or else the supplement, of ϕ; in either case its tangent would be the negative of that given by formula [14].

If the equations of the lines had been given in the form:

$$A_1x + B_1y + C_1 = 0, \qquad (3)$$

and $$A_2x + B_2y + C_2 = 0, \qquad (4)$$

then $m_1 = -\frac{A_1}{B_1}$, $m_2 = -\frac{A_2}{B_2}$, and formula [14] becomes

$$\tan\phi = \frac{-\frac{A_1}{B_1} + \frac{A_2}{B_2}}{1 + \frac{A_1}{B_1}\frac{A_2}{B_2}} = \frac{A_2B_1 - A_1B_2}{A_1A_2 + B_1B_2}. \qquad [15]$$

EXERCISES

Find the tangent of the angle from the first line to the second in each of the following cases, and draw the figures:

1. $3x - 4y - 7 = 0, \quad 2x - y - 3 = 0;$

2. $5x + 12y + 1 = 0, \quad x - 2y + 6 = 0;$

3. $2x = 3y + 9, \quad 6y = 4x + 2;$

4. $\frac{x}{a} + \frac{y}{b} = 1, \quad \frac{x}{a} - \frac{y}{b} = 1;$

5. $x\cos\alpha + y\sin\alpha = p, \quad \frac{x}{a} + \frac{y}{b} = 1.$

62. Condition that two lines are parallel or perpendicular. From formula [14] can be seen at once the relations that must hold between m_1 and m_2 if the lines (1) and (2) (Art. 61) are parallel or perpendicular. If these lines are parallel, then $\phi = 0$, and therefore $\tan\phi = 0$;

hence
$$\frac{m_1 - m_2}{1 + m_1 m_2} = 0,$$

i.e.,
$$m_1 = m_2,$$

which is the condition that lines (1) and (2) are parallel.* This condition is also evident from a mere inspection of equations (1) and (2).

If the lines (1) and (2) (Art. 61) are perpendicular, then $\phi = 90°$ and $\tan\phi = \infty$,

i.e.,
$$\frac{m_1 - m_2}{1 + m_1 m_2} = \infty, \text{ hence } 1 + m_1 m_2 = 0,$$

i.e.,
$$m_2 = -\frac{1}{m_1},$$

which is the condition that (1) and (2) are perpendicular.

So also from [15] the lines

$$A_1x + B_1y + C_1 = 0 \quad \text{and} \quad A_2x + B_2y + C_2 = 0$$

are parallel if (and only if) $A_2B_1 - A_1B_2 = 0$,

i.e., if
$$A_1 : B_1 = A_2 : B_2;$$

and they are perpendicular if (and only if) $A_1A_2 + B_1B_2 = 0$,

i.e., if
$$A_1 : B_1 = -B_2 : A_2.$$

The condition just found enables one to write down readily the equations of lines that are parallel or perpendicular to given lines, and which also pass through given points.

---

* It must not be forgotten that this conclusion is drawn only for lines that are *not* perpendicular to the x-axis; because if the lines are perpendicular to the x-axis then equations (1) and (2) are inapplicable (cf. Art. 56).

E.g., let it be required to write the equation of a line that is parallel to the line

$$y = 3x + 7. \qquad . \quad . \quad . \quad (1)$$

The slope of this line is 3, hence any other line whose slope is 3 is parallel to the given line,

i.e., $$y = 3x + b, \qquad . \quad . \quad . \quad (2)$$

is, for all values of b, parallel to line (1).

If it be required that the line (2) shall also pass through a given point, (1, 5) for example, it is only necessary to determine rightly the value of b. This is done by remembering that if the line (2) passes through the point (1, 5), then these coördinates must satisfy equation (2),

i.e., $$5 = 3 \cdot 1 + b, \text{ whence } b = 2.$$

Therefore the line $y = 3x + 2$ is not only parallel to the line $y = 3x + 7$, but also passes through the point (1, 5).

Similarly $y = -\frac{1}{3}x + b$, whatever the value of b, is perpendicular to $y = 3x + 7$.

Again, the line $3x + 5y + k = 0$, whatever the value of k, is parallel to the line $3x + 5y - 15 = 0$; and the line $5x - 3y + k = 0$ is perpendicular to $3x + 5y - 15 = 0$. Here again the arbitrary constant k may be so determined that this line shall pass through any given point. So also the lines $A_1x + B_1y + C_1 = 0$ and $A_1x + B_1y + C_2 = 0$ are parallel, while $A_1x + B_1y + C_1 = 0$ and $B_1x - A_1y + C_2 = 0$ are perpendicular to each other.

This condition for parallelism and for perpendicularity of two lines may also be stated thus: *two lines are parallel if their equations differ* (or may be made to differ) *only in their constant terms; two lines are perpendicular if the coefficients of x and y in the one are equal* (or can be made equal), *respectively, to the coefficients of $-y$ and x in the other.*

EXERCISES

1. Write down the equations of the set of lines parallel to the lines:

$$(\alpha)\quad y = 6x - 2;\qquad (\beta)\quad 3x - 7y = 3;$$

$$(\gamma)\quad x\cos 30^\circ + y\sin 30^\circ = 8;\qquad (\delta)\quad \frac{x}{2} - \frac{y}{3} = 1.$$

2. Explain why it is that the constant term in the answers to Ex. 1 is left undetermined or arbitrary.

3. Find the tangent of the angle between the lines (α) and (β) in Ex. 1; also for the lines (β) and (δ), and (α) and (δ) of Ex. 1.

4. Write the equations of lines perpendicular to those given in Ex. 1.

5. By the method of Art. 62 find the equation of the line that passes through the point $(^-9, 1)$, and is parallel to the line $y = 6x - 2$.

6. Solve Ex. 4 by means of equation [11], Art. 53.

7. Find the equation of the line that is parallel to the line $Ax + By + C = 0$ and that passes through the point (x_1, y_1); make two solutions, one by the method of Ex. 4, and the other by Ex. 5.

Find the equation of the straight line

8. through the point $(2, {}^-5)$ and parallel to the line $y = 2x + 7$.

9. through the point $(^-1, {}^-1)$ and perpendicular to $y = 2x + 7$; solve by two methods.

10. through the point (0, 0) and parallel to the line

$$\frac{3}{2}x - \frac{7}{5}y = \frac{x - y + 1}{9}.$$

11. perpendicular to the line $2y + 7x - 1 = 0$, and passing through the point midway between the two points in which this line meets the coördinate axes.

12. Find the foot of the perpendicular from the origin to the line $5x - 7y = 2$.

63. Line which makes a given angle with a given line. The formula

$$\tan\phi = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2} \quad \text{(Art. 61)}$$

states the relation existing between the tangents of the angles θ_1, θ_2, and ϕ (see Fig. 47); hence if any two of these angles are known, this equation determines the value of the third. Thus this formula may be employed to determine the slope of a line that shall make a given angle with a given line.

E.g., given the line $3y - 5x + 7 = 0$, to find the equation of a line that shall make an angle of 60° with this line. Here $\phi = 60°$, *i.e.*, $\tan\phi = \sqrt{3}$, and if θ_1 be the angle which the given line makes with the x-axis, and θ_2 that made by the line whose equation is sought, then $\tan\theta_1 = \frac{5}{3}$. Substituting these values in the above formula, it becomes

$$\sqrt{3} = \frac{\frac{5}{3} - \tan\theta_2}{1 + \frac{5}{3}\tan\theta_2},$$

whence

$$\tan\theta_2 = \frac{5 - 3\sqrt{3}}{3 + 5\sqrt{3}}, \text{ and } y = \frac{5 - 3\sqrt{3}}{3 + 5\sqrt{3}} \cdot x + k$$

is the equation of a line fulfilling the required conditions, — k may be so determined that this line shall also pass through any given point.

It is to be remarked that through any given point there may be drawn *two* lines, each of which shall make, with a given line, an angle of any desired magnitude.

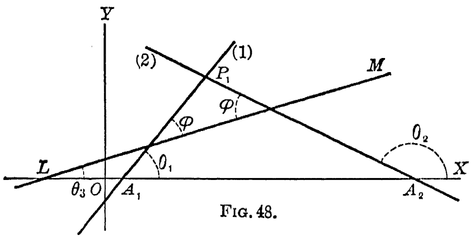


FIG. 48.

E.g., through $P_1 \equiv (x_1, y_1)$ the lines (1) and (2) may be so drawn that each shall make an angle ϕ with the given line *LM*. Let line (1) make an angle θ_1, line (2) an angle θ_2, and *LM* an angle θ_3, with the x-axis; then

$$\phi = \theta_1 - \theta_3, \text{ and } 180 - \phi = \theta_2 - \theta_3;$$

which gives

$$\tan\phi = \frac{\tan\theta_1 - \tan\theta_3}{1 + \tan\theta_1 \tan\theta_3}, \text{ and } -\tan\phi = \frac{\tan\theta_2 - \tan\theta_3}{1 + \tan\theta_2 \tan\theta_1}.$$

In these equations ϕ and θ_3 are known, hence $\tan\theta_1$ and $\tan\theta_2$ can be found. Having found $\tan\theta_1$ and $\tan\theta_2$ the equations of lines (1) and (2) may at once be written down, either by means of equation [11], or by the method employed in Art. 62.

EXERCISES

1. Find the equations of the two lines which pass through the point (5, 8), and each of which makes an angle of 45° with the line $2x - 3y = 6$.

2. Show that the equations of the two straight lines passing through the point $(3, -2)$ and inclined at 60° to the line $x\sqrt{3} + y = 1$ are

$$y + 2 = 0 \text{ and } y - x\sqrt{3} + 2 + 3\sqrt{3} = 0.$$

Find the equation of the straight line

3. making an angle of $-\frac{\pi}{4}$ with the line $3x - 4y = 7$; construct the figure. Why is there an undetermined constant in the resulting equation?

4. making an angle of $+60°$ with the line $5x + 12y + 1 = 0$; construct the figure.

5. making an angle of $-30°$ with the line $x - 2y + 1 = 0$, and passing through the point (1, 3); making an angle of $+30°$, and passing through the same point.

6. making an angle of $\pm 135°$ with the line $x + y = 2$, and passing through the origin.

7. making the angle $\tan^{-1}\frac{b}{a}$ with the line $\frac{x}{a} + \frac{y}{b} = 1$, and passing through the point $\left(\frac{a}{2}, \frac{b}{2}\right)$.

8. Find the equation of a line through the point (4, 5) forming with the lines $2x - y + 3 = 0$ and $3y + 6x = 7$ a right-angled triangle. Find the vertices of the triangle (two solutions).

9. Show that the triangle whose vertices are the points $(2, 1)$, $(3, -2)$, $(-4, -1)$ is a right triangle.

10. Prove analytically that the perpendiculars erected at the middle points of the sides of the triangle, the equations of whose sides are

$$x + y + 1 = 0,\ 3x + 5y + 11 = 0, \text{ and } x + 2y + 4 = 0,$$

meet in a point which is equidistant from the vertices.

11. Find the equations of the lines through the vertices and perpendicular to the opposite sides of the triangle in exercise 10. Prove that these lines also meet in a common point.

12. A line passes through the point $(2, -3)$ and is parallel to the line through the two points $(4, 7)$ and $(-1, -9)$; find its equation.

13. Find the equation of the line which passes through the point of intersection of the two lines $10x + 5y + 11 = 0$, and $x + 2y + 14 = 0$, and which is perpendicular to the line $x + 7y + 1 = 0$.

This problem may be solved by first finding the point of intersection $(\frac{16}{5}, -\frac{43}{5})$ of the two given lines, and then, by formula [11] (see also Art. 62), writing the equation of the required line, viz.:

$$y + \tfrac{43}{5} = 7\left(x - \tfrac{16}{5}\right),$$

which reduces to

$$7x - y = 31.$$

The problem may also be solved somewhat more briefly, and much more elegantly, by employing the theorem of Art. 41. By this theorem the equation of the required line is of the form

$$10x + 5y + 11 + k(x + 2y + 14) = 0,$$

i.e.,

$$(10 + k)x + (5 + 2k)y + 11 + 14k = 0.$$

It only remains to determine the constant k, so that this line shall be perpendicular to $x + 7y + 1 = 0$. By Art. 62 its slope must be $-\dfrac{1}{-\frac{1}{7}} = 7$, hence $-\dfrac{10 + k}{5 + 2k} = 7$, whence $k = -3$.

Substituting this value of k above, the required equation becomes $7x - y = 31$, as before.

14. By the second method of exercise 13 find the equation of the line which passes through the point of intersection of the two lines $2x + y = 5$ and $x = 3y - 8$, and which is: (1) parallel to the line $4y = 6x + 1$; (2) perpendicular to this line; (3) inclined at an angle of 60° to this line; (4) passes through the point $(-1, 3)$.

15. Solve exercise 10 by the method of exercise 14.

16. Do the lines $2x+3y=13$, $5x-y=7$, and $x-4y+10=0$ meet in a common point? What are the angles they make with each other?

17. Find the angles of the triangle of exercise 10.

18. When are the lines

$$x+(a+b)y+c=0 \text{ and } a(x+ay)+b(x-by)+d=0$$

parallel? when perpendicular?

19. Find the value of p for each of the two parallel lines

$$y=3x+7 \text{ and } y=3x-5;$$

and hence find the distance between these lines [cf. Art. 58 (3) and (4)].

20. What is the distance between the two parallel lines

$$5x-3y+6=0 \text{ and } 6y-10x=7?$$

21. Find the cosine of the angle between the lines

$$y-4x+8=0 \text{ and } y-6x+9=0.$$

22. What relation exists between the two lines

$$y=3x+7 \text{ and } y=-3x-3?$$

23. Find the angle between the two straight lines $3x=4y+7$ and $5y=12x+6$; and also the equations of the two straight lines which pass through the point (4, 5) and make equal angles with the two given lines.

24. Find the angle between the two lines

$$3x+y+12=0 \text{ and } x+2y-1=0.$$

Find also the coördinates of their point of intersection, and the equations of the lines drawn perpendicular to them from the point $(3, {}^{-}2)$.

64. The distance of a given point from a given line. This problem is easily solved for any particular case thus: find the equation of the line which passes through the given point and which is parallel to the given line (Art. 62), then find the distance (p) from the origin to each of these two lines [Art. 58, (3) and (4)], and finally subtract one of these distances from the other; the result is the distance between the given line and the given point.

E.g., find the distance of the point $P_1 \equiv (2, \frac{3}{2})$ from the line

$$3x + 4y - 7 = 0. \quad . \quad . \quad . \qquad (1)$$

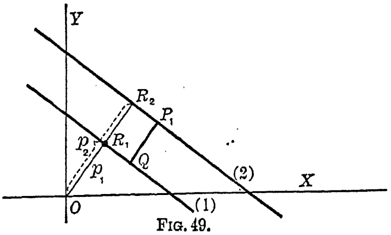


FIG. 49.

Let line (1) be the locus of equation (1), and P_1 be the given point. Through P_1 draw the line (2) parallel to line (1), also draw QP_1 perpendicular to line (1), $OR_1 (= p_1)$ perpendicular to line (1), and $OR_2 (= p_2)$ perpendicular to line (2). Then $d = QP_1 = p_2 - p_1$.

The equation of a line parallel to line (1) is of the form $3x + 4y + k = 0$; this will represent line (2) itself if k be so determined that the line shall pass through the point

$$P_1 \equiv (2, \tfrac{3}{2}),\ i.e., \text{ if } 3 \cdot 2 + 4 \cdot \tfrac{3}{2} + k = 0,\ i.e., \text{ if } k = -12.$$

The equation of line (2) is then

$$3x + 4y - 12 = 0 \quad . \quad . \quad . \qquad (2)$$

Therefore [by Art. 58, (3) or (4)]

$$p_2 = \frac{12}{+\sqrt{4^2 + 3^2}} = \frac{12}{5}, \text{ and } p_1 = \frac{7}{+\sqrt{4^2 + 3^2}} = \frac{7}{5};$$

hence the required distance is $d = QP_1 = \dfrac{12 - 7}{5} = 1$.

Similarly, in general, to find the distance of any given point $P_1 \equiv (x_1, y_1)$ from any given line

$$Ax + By + C = 0 \quad . \quad . \quad . \qquad (1)$$

let line (1) be the locus of equation (1) and let P_1 be the given point. The equation of a line parallel to (1) is of the form $Ax + By + K = 0$; this will be the line (2) if

$Ax_1 + By_1 + K = 0$, *i.e.*, if $K = -(Ax_1 + By_1)$. The equation of line (2) is then

$$Ax + By - (Ax_1 + By_1) = 0. \quad . \quad . \quad . \quad (2)$$

Therefore $p_2 = \dfrac{Ax_1 + By_1}{\sqrt{A^2 + B^2}}$, $p_1 = \dfrac{-C}{\sqrt{A^2 + B^2}}$,

wherein the sign of the radical is to be chosen in accord with Art. 58 (3);

hence $$d = \frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}}. \quad . \quad . \quad . \quad . \quad [16]$$

If the equation of the given line is so written that its second member is zero, this formula may be translated into words thus: *To get the distance of a given point from a given line, write the first member of the equation alone, substitute for the variables therein the coördinates of the given point, and divide the result by the square root of the sum of the squares of the coefficients of x and y in the equation,* — the sign of this square root being chosen opposite to that of the number represented by C.

If, in formula [16], d is positive, then $p_2 > p_1$, and P_1 and the origin are on *opposite* sides of the given line; if d is negative, $p_2 < p_1$, and P_1 and the origin are on the *same* side of the given line.

EXERCISES

1. Find the distance of the point $(2, -7)$ from the line $3x - 6y + 1 = 0$.

By formula [16], $d = \dfrac{3 \cdot 2 - 6(-7) + 1}{-\sqrt{3^2 + 6^2}} = -\dfrac{49}{3\sqrt{5}}$.

This result, besides giving the numerical value of the distance, shows also that the point $(2, -7)$ and the origin are on the same side of the line $3x - 6y + 1 = 0$.

2. Find the distance of the point $(4, 5)$ from the line $4y + 5x = 20$.

3. Find the distance of the point $(2, 7)$ from the line $3y - 2x = 17$.

4. Find the distance of the point (a, b) from the line $\frac{x}{a}+\frac{y}{b}=1$.

5. Find the distance of the intersection of the two lines, $y+4=3x$ and $5x=y-2$, from the line $2y-7=9$. On which side of the latter line is the point?

6. Find the distance of the point of intersection of the lines $2x-5y=11$ and $4x=3y+15$ from the line $\frac{1}{2}x+\frac{y-5}{4}=6$. On which side of the latter line is the point? Plot the figure.

7. How far is the point $(^-6, ^-1)$ from $3y=7x+8$? On which side?

8. By the method of Art. 64, find the distance of the origin from the line $5x-2y=7$; also from the line $Ax+By+C=0$. Check the results by Art. 58 (3).

9. Find the distance of the point $(^-4, ^-5)$ from the line joining the two points $(3, ^-1)$ and $(^-4, 2)$. On which side is it?

10. Find the distance of the point (x_1, y_1) from the line $y=mx+b$.

11. Find the altitudes of the triangle formed by the lines whose equations are $x+y+1=0$, $3x+5y+11=0$, and $x+2y+4=0$. Check the result by finding the area of the triangle in two ways.

12. Show analytically that the locus of a point which moves so that the sum of its distances from two given straight lines is constant is itself a straight line.

13. Express by an equation that the point $P_1 \equiv (x_1, y_1)$ is equally distant from the two lines $2x-y=11$ and $4x=3y+5$. (Give two answers.) Should P_1 move in such a way as to be always equidistant from these two lines, what would be the equation of its locus?

14. Find, by the method of exercise 13, the equations of the bisectors of the angle formed by the lines $3x+4y=12$ and $4x+3y=24$.

65. Bisectors of the angles between two given lines. The bisector of an angle is the locus of a point which moves so that it is always equally distant (numerically) from the sides of the angle. From this property its equation may easily be found.

E.g., find the equations of the bisectors of the angles between the lines

$$3x+4y-1=0, \quad \ldots \quad (1)$$

and

$$12x-5y+6=0. \quad \ldots \quad (2)$$

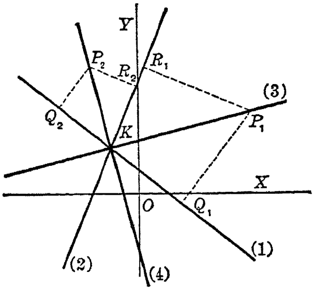


FIG. 50.

Let $P_1 \equiv (x_1, y_1)$ be any point on the bisector (3).

Then $Q_1P_1 \equiv -R_1P_1$ [since O and P_1 are on opposite sides of line (1) and on the same side of (2); or *vice versa*].

But $Q_1P_1 = \dfrac{3x_1 + 4y_1 - 1}{+\sqrt{3^2 + 4^2}}$

$= \dfrac{3x_1 + 4y_1 - 1}{5}$, (Art. 64),

and $R_1P_1 = \dfrac{12x_1 - 5y_1 + 6}{-\sqrt{12^2 + 5^2}} = \dfrac{12x_1 - 5y_1 + 6}{-13}$;

$$\therefore \frac{3x_1 + 4y_1 - 1}{5} = \frac{12x_1 - 5y_1 + 6}{13};$$

i.e., $21x_1 - 77y_1 + 43 = 0$. . . . (5)

Hence $21x - 77y + 43 = 0$ . . . (6)

is the equation of the bisector (3), for equation (5) asserts that if (x_1, y_1) be the coördinates of any point on this bisector they satisfy equation (6).

Similarly, let $P_2 \equiv (h, k)$ be any point on line (4), the other bisector, then $Q_2P_2 = R_2P_2$ [since O and P_2 are on opposite sides of the lines (1) and (2), or else both on the same side of each of these lines];

$$\therefore \quad \frac{3h + 4k - 1}{5} = -\frac{12h - 5k + 6}{13},$$

i.e., $99h + 27k + 17 = 0$. . . . (7)

Hence $99x + 27y + 17 = 0$ . . . (8)

is the equation of the bisector (4), for the same reason as given above.

Geometrically it is well known that two such bisectors, (3) and (4), are perpendicular to each other: their equations, also prove that fact.

The equations of the bisectors of the angles between any two lines, as $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$, are found in precisely the same way as that employed in the numerical example just considered.

EXERCISES

1. Find the equations of the bisectors of the angles between the two lines $x - y + 6 = 0$ and $\frac{3x-4}{2} = 5y - 7$.

2. Show that the line $11x + 3y + 1 = 0$ bisects one of the angles between the two lines $12x - 5y + 7 = 0$, and $3x + 4y - 2 = 0$. Which angle is it? Find the equation of the bisector of the other angle.

3. Show analytically that the bisectors of the interior angles of the triangle whose vertices are the points (1, 2), (5, 3), and (4, 7) meet in a common point.

4. Show analytically, for the triangle of Ex. 3, that the bisectors of one interior and the two opposite exterior angles meet in a common point.

5. Find the angle from the line $3x + y + 12 = 0$ to the line $ax + by + 1 = 0$, and also the angle from the line $ax + by + 1 = 0$ to the line $x + 2y - 1 = 0$.

By imposing upon a and b the two conditions: (1) that the angles just found are equal, and (2) that the line $ax + by + 1 = 0$ passes through the intersection of the other two lines, determine a and b so that this line shall be a bisector of one of the angles made by the other two given lines.

66. The equation of two lines. By the reasoning given in Art. 40, it is shown that if two straight lines are represented by the equations

$$A_1x + B_1y + C_1 = 0 \quad . \quad . \quad . \quad (1)$$

and

$$A_2x + B_2y + C_2 = 0, \quad . \quad . \quad . \quad (2)$$

then *both* these lines are represented by the equation

$$(A_1x + B_1y + C_1)(A_2x + B_2y + C_2) = 0; \quad . \quad . \quad . \quad (3)$$

i.e., two straight lines are here represented by an equation of the *second* degree.

Conversely, if an equation of the second degree, whose second member is zero, can have its first member separated into two first degree factors, with real coëfficients, as in equation (3), then its locus consists of two straight lines.

Thus the equation

$$2x^2 - xy - 3y^2 + 9x + 4y + 7 = 0$$

may be written in the form

$$(2x - 3y + 7)(x + y + 1) = 0,$$

which shows that it is satisfied when $2x - 3y + 7 = 0$, and also when $x + y + 1 = 0$. Its locus is therefore composed of the two lines whose equations are:

$$2x - 3y + 7 = 0, \text{ and } x + y + 1 = 0.$$

67. Condition that the general quadratic expression may be factored. The most general equation of the second degree between two variables may be written in the form

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0. \quad . \quad . \quad . \quad (1)$$

It is required to find the relation that must exist among the coëfficients of this equation in order that its first member may be separated into two rational factors, each of the first degree, *i.e.*, it is required to find the condition that the equation may be written thus:

$$(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0. \quad . \quad . \quad . \quad (2)$$

Evidently if equation (1) can be written in the form of equation (2), then the values of x obtained from equation (1) are rational, and are either

$$x = \frac{-c_1 - b_1y}{a_1} \quad \text{or} \quad x = \frac{-c_2 - b_2y}{a_2}.$$

Solving equation (1) for x in terms of y, by completing the square of the x-terms, it becomes

$$A^2x^2 + 2A(Hy + G)x + (Hy + G)^2$$
$$= -ABy^2 - 2AFy - AC + (Hy + G)^2,$$

i.e.,

$$Ax + Hy + G$$
$$= \sqrt{(H^2 - AB)y^2 - 2(HG - AF)y + G^2 - AC},$$

and finally,

$$x = -\frac{H}{A}y - \frac{G}{A} \pm \frac{1}{A}\sqrt{(H^2 - AB)y^2 - 2(HG - AF)y + G^2 - AC}.$$

But since x is, by hypothesis, expressible rationally in terms of y, therefore the expression under the radical sign is a perfect square, and therefore

$$(HG - AF)^2 - (H^2 - AB)(G^2 - AC) = 0,$$

i.e.,

$$\mathbf{ABC + 2FGH - AF^2 - BG^2 - CH^2 = 0.} \quad . \quad . \quad [17]$$

If this condition among the coëfficients is fulfilled, then equation (1) has for its locus two straight lines.

The expression $ABC + 2FGH - AF^2 - BG^2 - CH^2$ is called the **discriminant** of the quadratic, and is usually represented by the symbol Δ.

Note. The analytic work just given fails if $A = 0$. In that case equation (1) may be solved for y instead of solving it for x, and the same condition, viz. $\Delta = 0$, results. If, however, *both* A and B are zero, then the above method fails altogether. In that case equation (1) reduces to

$$2Hxy + 2Gx + 2Fy + C = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If the first member of equation (3) can be factored, then evidently the equation must take the form

$$(ax + b)(cy + d) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which shows that equation (3) is satisfied for all values of y provided $x = -\frac{b}{a}$, a constant. Let $-\frac{b}{a}$ be represented by k, then equation (4) becomes

$$2Hky + 2Gk + 2Fy + C = 0,$$

i.e.,

$$2(Hk + F)y + 2Gk + C = 0,$$

and is satisfied for all values of y;

$$\therefore \qquad Hk + F = 0, \text{ and } 2Gk + C = 0;$$

hence, eliminating k, $$2FG - CH = 0.$$

But this is the expression to which Δ reduces when $A = B = 0$ and $H \neq 0$; hence, in all cases, $\Delta = 0$ is the *necessary* condition that the above quadratic may be factored.

That $\Delta = 0$ is also the *sufficient* condition is readily seen by retracing the steps from equation [17] when at least one of the coëfficients A, B differs from zero. But it is also sufficient when $A = B = 0$; for, in that case, $\Delta = 0$ becomes $2FG - CH = 0$, which may be written $\frac{F}{H}\cdot\frac{G}{H} = \frac{C}{2H}$. Under the same circumstances equation (1) becomes equation (3), which may be written

$$xy + \frac{F}{H}x + \frac{G}{H}y + \frac{C}{2H} = 0. \quad \dots\dots\dots \quad (4)$$

Substituting $\frac{F}{H}\cdot\frac{G}{H}$ for $\frac{C}{2H}$ in equation (4), it becomes

$$xy + \frac{F}{H}x + \frac{G}{H}y + \frac{F}{H}\cdot\frac{G}{H} = 0 \quad \dots\dots\dots \quad (5)$$

i.e., $$\left(x + \frac{G}{H}\right)\left(y + \frac{F}{H}\right) = 0,$$

which establishes the sufficiency of the condition for this case also.

To illustrate the use of equation [17]* examine the equation of Art. 66:

$$2x^2 - xy - 3y^2 + 9x + 4y + 7 = 0.$$

---

* As an illustration of another practical method of factoring a quadratic expression, *when factoring is possible*, *i.e.*, if equation [17] holds, find the factors of

$$2x^2 - 7xy - 15y^2 + 7x - 17y - 4.$$

Factor the terms free from y,

$$2x^2 + 7x - 4 \equiv (2x - 1)(x + 4);$$

factor the terms free from x,

$$-15y^2 - 17y - 4 \equiv (3y - 1)(-5y + 4);$$

combine the factors containing the same constant term,

$$(2x + 3y - 1), \quad (x - 5y + 4);$$

these will be the factors of the given quadratic expression.

Here $A = 2,\ B = -3,\ C = 7,\ H = -\frac{1}{2},\ G = \frac{9}{2}$, and $F = 2$;

hence $$\Delta = -42 - 9 - 8 + \frac{243}{4} - \frac{7}{4} = 0;$$

therefore the first member can be factored.

The factors may be found as follows: transposing, dividing by 2, and completing the square of the x-terms, the equation may be written in the form $$x^2 + \frac{9-y}{2}x + \left(\frac{9-y}{4}\right)^2 = \frac{25}{16}(y^2 - 2y + 1);$$

i.e., $$\left(x + \frac{9-y}{4}\right)^2 = \left\{\frac{5}{4}(y-1)\right\}^2;$$

therefore the given equation, divided by 2, may be written in the form, $$\left(x + \frac{9-y}{4}\right)^2 - \left\{\frac{5}{4}(y-1)\right\}^2 = 0;$$

i.e., $$\left\{\left(x + \frac{9-y}{4}\right) + \frac{5}{4}(y-1)\right\}\left\{\left(x + \frac{9-y}{4}\right) - \frac{5}{4}(y-1)\right\} = 0,$$

i.e., $$(x + y + 1)(x - \tfrac{3}{2}y + \tfrac{7}{2}) = 0;$$

hence the locus of the original equation consists of the straight lines $$x + y + 1 = 0 \text{ and } 2x - 3y + 7 = 0,$$

which agrees with the result of Art. 66.

EXERCISES

Prove that the following equations represent pairs of straight lines; find in each case the equations of the two lines, the coördinates of their point of intersection, and the angle between them.

1. $6y^2 - xy - x^2 + 30y + 36 = 0$.

2. $x^2 - 2xy - 3y^2 + 2x - 2y + 1 = 0$.

3. $x^2 - 2xy\sec\alpha + y^2 = 0$.

4. $x^2 + 6xy + 9y^2 + 4x + 12y - 5 = 0$.

5. For what value of k will the equation $$x^2 - 3xy + y^2 + 10x - 10y + k = 0$$ represent two straight lines?

Suggestion: Place the discriminant (Δ) equal to zero, and thus find $k = 20$.

Find the values of k for which the following equations represent pairs of straight lines. Find also the equation of each line, the point of intersection of each pair of lines, and the angle between them.

6. $6x^2 + 2kxy + 12y^2 + 22x + 31y + 20 = 0$.

7. $12x^2 + 36xy + ky^2 + 6x + 6y + 3 = 0$.

8. $4x^2 - 12xy + 9y^2 - kx + 6y + 1 = 0$.

9. The equations of the opposite sides of a parallelogram are

$$x^2 - 7x + 6 = 0 \text{ and } y^2 - 14y + 40 = 0.$$

Find the equations of the diagonals.

10. Find the conditions that the straight lines represented by the equation $Ax^2 + 2Bxy + Cy^2 = 0$ may be real; imaginary; coincident; perpendicular to each other.

11. Show that $6x^2 + 5xy - 6y^2 = 0$ is the equation of the bisectors of the angles made by the lines $2x^2 + 12xy + 7y^2 = 0$. Does the first set of lines fulfil the test of exercise 10 for perpendicularity?

68. Equations of straight lines: coördinate axes oblique. Since in the derivation of equations [9] and [10] (Arts. 51 and 52) only properties of similar triangles were employed, therefore these two equations are true whether the coördinate axes are rectangular or oblique.

The other three standard forms however, viz. $y = mx + b$, $y - y_1 = m(x - x_1)$, and $x \cos a + y \sin a = p$, the derivation of which depends upon *right* triangles, are no longer true if the axes are inclined to each other at an angle $\omega \neq \frac{\pi}{2}$. Equations which correspond to these, but which are referred to oblique axes, will now be derived.

(1) *Equation of straight line through a given point and in a given direction.* Let LL_1 be the straight line through the fixed point $P_1 \equiv (x_1, y_1)$ and making an angle θ with the x-axis, let $P \equiv (x, y)$ be any other point on LL_1, and let ω be the angle between the axes.

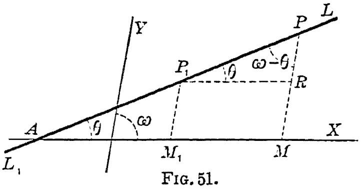


FIG. 51.

Draw P_1R parallel to the x-axis, also draw the ordinates M_1P_1 and MP. Then

$$\theta = \angle XAL = \angle RP_1L \quad \text{and} \quad \angle P_1PR = \omega - \theta.$$

Hence $$\frac{RP}{P_1R}=\frac{\sin\theta}{\sin(\omega-\theta)}. \qquad \text{[law of sines]}$$

Substituting in this equation the coördinates of P_1 and P, it becomes

$$\frac{y-y_1}{x-x_1}=\frac{\sin\theta}{\sin(\omega-\theta)},$$

i.e., $$y-y_1=\frac{\sin\theta}{\sin(\omega-\theta)}(x-x_1), \quad . \quad . \quad . \quad [18]$$

which is the required equation.

When $\omega=\frac{\pi}{2}$ this equation reduces to equation [11], *i.e.*, to $y-y_1=m(x-x_1)$, where $m=\tan\theta$; but it must be observed that if $\omega\neq\frac{\pi}{2}$, then the coefficient of x in equation [18] does not represent the *slope* of the line. If, however, the slope of the line [18], *i.e.*, the $\tan\theta$ for this line, is desired, it is easily found thus: let $\frac{\sin\theta}{\sin(\omega-\theta)}=k$, from which is obtained $\tan\theta=\frac{k\sin\omega}{1+k\cos\omega}$.

If, in the derivation of equation [18], the given point is that in which the line LL_1 meets the y-axis, *i.e.*, if $P_1\equiv(0,b)$, then equation [18] reduces to

$$y=\frac{\sin\theta}{\sin(\omega-\theta)}x+b, \quad . \quad . \quad . \quad [19]$$

which corresponds to equation [12], but the coefficient of x is not the slope of the line.

(2) *Equation of a straight line in terms of the perpendicular upon it from the origin, and the angles which this perpendicular makes with the axes.*

Let LL_1 be the straight line whose equation is sought, let the perpendicular from the origin upon it ($ON = p$) make the angles α and β respectively with the axes,* and let $P \equiv (x, y)$ be any point on LL_1.

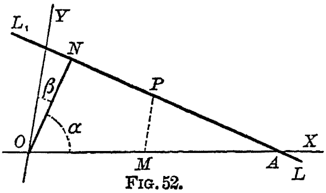


FIG. 52.

Draw the ordinate MP; then, by Art. 17,

$$OM \cos \alpha + MP \cos \beta = ON,$$

i.e.,

$$x \cos \alpha + y \cos \beta = p, \quad . \quad . \quad . \quad [20]$$

which is the required equation.

If ω is the angle between the axes, then $\beta = \omega - \alpha$, and equation [20] may be written $x \cos \alpha + y \cos (\omega - \alpha) = p$. If $\omega = \frac{\pi}{2}$, then this equation reduces to $x \cos \alpha + y \sin \alpha = p$, which agrees with equation [13].

EXERCISES

1. The axes being inclined at the angle 60°, find the inclination of the line $y = 2x + 5$ to the x-axis.

2. The axes being inclined at the angle $\frac{\pi}{4}$, find the angles at which the lines $3y + 7x - 1 = 0$ and $x + y + 2 = 0$ cross the x-axis.

3. Find the angle between the lines in exercise 2.

4. The center of an equilateral triangle of side 6 is joined by straight lines to the vertices. If two of these lines are taken as coördinate axes, find the coördinates of the vertices, and the equations of the sides.

5. Prove that for every value of ω, the lines $x + y = c$ and $x - y = d$ are perpendicular to each other.

---

* The angles α and β are the *direction angles* of the line ON, and their cosines are the *direction cosines* of that line.

69. Equations of straight lines: polar coördinates.

(1) *Line through two given points.* Let OR be the initial line, O the pole, $P_1 \equiv (\rho_1, \theta_1)$, and $P_2 \equiv (\rho_2, \theta_2)$, the two given points, and let $P \equiv (\rho, \theta)$ be any other point on the line through P_1 and P_2.

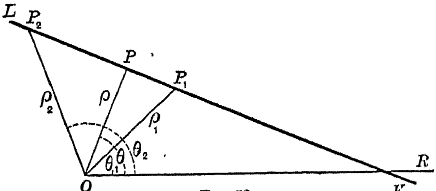


FIG. 53.

Then (if $\triangle$ stands for 'area of triangle')

$$\triangle\, OP_1P_2 = \triangle\, OP_1P + \triangle\, OPP_2,$$

i.e., $\frac{1}{2}\rho_1\rho_2 \sin(\theta_2-\theta_1) = \frac{1}{2}\rho\rho_1 \sin(\theta-\theta_1) + \frac{1}{2}\rho_2\rho\sin(\theta_2-\theta)$,

hence
$$\rho\rho_1 \sin(\theta-\theta_1) + \rho_1\rho_2 \sin(\theta_1-\theta_2) + \rho_2\rho\sin(\theta_2-\theta) = 0.^* \quad . \quad . \quad . \quad [21]$$

This equation may also be written in the form

$$\frac{\sin(\theta_1-\theta_2)}{\rho} + \frac{\sin(\theta_2-\theta)}{\rho_1} + \frac{\sin(\theta-\theta_1)}{\rho_2} = 0.^*$$

(2) *Equation of the line in terms of the perpendicular upon it from the pole, and the angle which this perpendicular makes with the initial line.* Let OR be the initial line, O the pole, and LK the line whose equation is sought. Also, let $N \equiv (p, \alpha)$ be the foot of the perpendicular from O upon LK, and let $P \equiv (\rho, \theta)$ be any other point on LK. Draw ON and OP; then

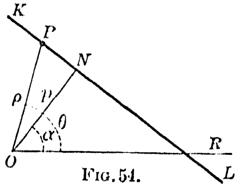


FIG. 54.

$$\frac{ON}{OP} = \cos NOP,$$

i.e.,
$$\rho\cos(\theta-\alpha) = p, \quad . \quad . \quad . \quad . \quad [22]$$

which is the required equation.

---

* Observe the symmetry here; cf. foot-note, Art. 29.

EXERCISES

1. Construct the lines:

(*a*) $\rho\cos(\theta - 30°) = 10$; (*c*) $\rho\cos\left(\theta - \frac{\pi}{4}\right) = 9$;

(*b*) $\rho\sin\theta = 2$; (*d*) $\rho\cos(\theta - \pi) = 6$.

2. Find the polar equations of straight lines at a distance 3 from the pole, and: (1) parallel to the initial line; (2) perpendicular to the initial line.

3. A straight line passes through the points $(5, {}^{-}45°)$ and $(2, 90°)$; find its polar equation.

4. Find the polar equation of a line passing through a given point (ρ_1, θ_1) and cutting the initial line at a given angle $\phi = \tan^{-1} k$.

5. Find the polar coördinates of the point of intersection of the lines

$$\rho\cos\left(\theta - \frac{\pi}{2}\right) = 2a, \qquad \rho\cos\left(\theta - \frac{\pi}{6}\right) = a.$$

EXAMPLES ON CHAPTER V

1. The points $({}^{-}1, 2)$ and $(3, {}^{-}2)$ are the extremities of the base of an equilateral triangle. Find the equations of the sides, and the coördinates of the third vertex. Two solutions.

2. Three of the vertices of a parallelogram are at the points $(1, 1)$, $(3, 4)$, and $(5, {}^{-}2)$. Find the fourth vertex. (Three solutions.) Find also the area of the parallelogram.

3. Find the equations of the two lines drawn through the point $(0, 3)$, such that the perpendiculars let fall from the point $(6, 6)$ upon them are each of length 3.

4. Perpendiculars are let fall from the point $(5, 0)$ upon the sides of the triangle whose vertices are at the points $(4, 3)$, $({}^{-}4, 3)$, and $(0, {}^{-}5)$. Show that the feet of these three perpendiculars lie on a straight line.

Find the equation of the straight line

5. through the origin and the point of intersection of the lines $x - y = 4$ and $7x + y + 20 = 0$. Prove that it is a bisector of the angle formed by the two given lines.

6. through the intersection of the lines $3x - 4y + 1 = 0$ and $5x + y = 1$, and cutting off equal intercepts from the axes.

7. through the point $(1, 2)$, and intersecting the line $x + y = 4$ at a distance $\frac{1}{3}\sqrt{6}$ from this point.

8. Find the equation of a straight line through the point (4, 5) and making equal angles with the lines $3x = 4y + 7$ and $5y = 12x + 6$.

9. Prove analytically that the diagonals of a square are of equal length, bisect each other, and are at right angles.

10. Prove analytically that the line joining the middle points of two sides of a triangle is parallel to the third side and equal to half its length.

11. Find the locus of the vertex of a triangle whose base is $2a$ and the difference of the squares of whose sides is $4c^2$. Trace the locus.

12. Find the equations of the lines from the vertex (4, 3) of the triangle of Ex. 4, trisecting the opposite side. What are the ratios of the areas of the resulting triangles?

13. A point moves so that the sum of its distances from the lines $y - 3x + 11 = 0$ and $7x - 2y + 1 = 0$ is 6. Find the equation of its locus. Draw the figure.

14. Find the equation of the path of the moving point of Ex. 13, if the distances from the fixed lines are in the ratio 3 : 4.

15. Solve examples 13 and 14, taking the given lines as axes.

16. The point (2, 9) is the vertex of an isosceles right triangle whose hypotenuse is the line $3x - 7y = 2$. Find the other vertices of the triangle.

17. The axes of coördinates being inclined at the angle 60°, find the equation of a line parallel to the line $x + y = 3a$, and at a distance $\frac{a\sqrt{3}}{2}$ from it.

18. Find the point of intersection of the lines

$$\rho = \frac{2a}{\cos\left(\theta - \frac{\pi}{2}\right)} \quad \text{and} \quad \rho \cos\left(\theta - \frac{\pi}{6}\right) = a.$$

For what value of θ, in each line, is $\rho = \infty$? At what angles do these lines cut their polar axes? Find the angle between the lines. Plot these lines.

19. Find the equation of a straight line through the intersection of $y = 7x - 4$ and $2x + y = 5$, and forming with the x-axis the angle $\frac{\pi}{3}$.

20. Find the equation of the locus of a point which moves so as to be always equidistant from the points (2, 1) and (−3, −2).

21. Find the equation of the locus of a point which moves so as to be always equidistant from the points $(0, 0)$ and $(3, 2)$. Show that the points $(0, 0)$, $(3, 2)$, and $(1, {}^{-}1)$ are the vertices of an isosceles triangle.

22. Find the center and radius of the circle circumscribed about the triangle whose vertices are the points $(2, 1)$, $(3, {}^{-}2)$, $({}^{-}4, {}^{-}1)$.

23. Find analytically the equation of the locus of the vertex of a triangle having its base and area constant.

24. Prove analytically that the locus of a point equidistant from two given points (x_1, y_1) and (x_2, y_2) is the perpendicular bisector of the line joining the given points.

25. The base of a triangle is of length 5, and is given in position; the difference of the squares of the other two sides is 7; find the equation of the locus of its vertex.

26. What lines are represented by the equations:

$$(\alpha)\quad x^2y = xy^2;\qquad (\beta)\quad 14x^2 - 5xy - y^2 = 0;\qquad (\gamma)\quad xy = 0?$$

27. What must be the value of c in order that the lines $3x + y - 2 = 0$, $2x - y - 3 = 0$, and $5x + 2y + c = 0$ shall pass through a common point?

28. By finding the area of the triangle formed by the three points $(3a, 0)$, $(0, 3b)$ and $(a, 2b)$, prove that these three points are in a straight line. Prove this also by showing that the third point is on the line joining the other two.

29. Find, by the method of Art. 39, the point of intersection of the two lines $2x - 3y + 7 = 0$ and $4x = 6y + 2$; and interpret the result by means of Arts. 41 and 60.

30. Prove by Art. 10 (cf. also Arts. 41 and 60), that the equations of two parallel lines differ only in the constant term.

31. Find the equations of two lines each drawn through the point $(4, 3)$, and forming with the axes a triangle whose area is 8.

32. Find the equation of a line through the point $(2, {}^{-}5)$, such that the portion between the axes is divided by the given point in the ratio $7:5$.

33. Find the equation of the perpendicular erected at the middle point of the line joining $(5, 2)$ to the intersection of the two lines

$$x + 2y = 11 \quad\text{and}\quad 9x - 2y = 59.$$

34. A point moves so that the square of its distance from the origin equals twice the square of its distance from the x-axis; find the equation of its locus.

35. Given the four lines

$x - 2y + 2 = 0$, $x + 2y - 2 = 0$, $3x - y - 3 = 0$ and $x + y + 6 = 0$;

these lines intersect each other in six points; find the equations of the three new lines (diagonals), each of which is determined by a pair of the above six points of intersection.

36. Find the points of intersection of the loci:

$$(\alpha) \quad \rho \cos\left(\theta - \frac{\pi}{3}\right) = a \text{ and } \rho \cos\left(\theta - \frac{\pi}{6}\right) = a;$$

$$(\beta) \quad \rho \cos\left(\theta - \frac{\pi}{2}\right) = \frac{3a}{4} \text{ and } \rho = a \sin \theta.$$

If two sides of a triangle are taken as axes, the vertices are $(0, 0)$, $(x_1, 0)$, $(0, y_2)$. Prove analytically that:

37. the medians of a triangle meet in a point;

38. the perpendicular from each vertex to the opposite sides meet in a point;

39. the line joining the middle points of two sides of a triangle is parallel to the third side.

40. Show that the equation $56x^2 - 441xy - 56y^2 - 79x - 47y + 9 = 0$ represents the bisectors of the angles between the straight lines represented by $15x^2 - 16xy - 48y^2 - 2x + 16y - 1 = 0$.

41. Two lines are represented by the equation

$$Ax^2 + 2Hxy + By^2 = 0.$$

Find the angle between them.

CHAPTER VI

TRANSFORMATION OF COÖRDINATES

70. That the coördinates of a point which remains fixed in a plane are changed by changing the axes to which this fixed point is referred, is an immediate consequence of the definition of coördinates.

It is also evident that the different *kinds* of coördinates of any given point (Cartesian and polar, for example) are connected by definite relations if the elements of reference (the axes) are related in position. *E.g.*, the point Q, when referred to the polar axis OX and the pole O, has the coördinates (5, 30°), but when it is referred to the rectangular axes OX and OY the coördinates of this same point are $(\frac{5}{2}\sqrt{3}, \frac{5}{2})$; and generally, if (ρ, θ) be the coördinates of a point when referred to OX and O, then $(\rho\cos\theta, \rho\sin\theta)$ are its coördinates when it is referred to the rectangular axes OX and OY.

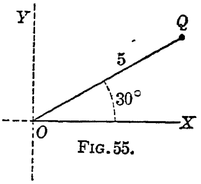


FIG. 55.

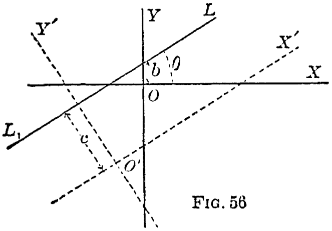


FIG. 56

Again: while a *curve* remains fixed in a plane, its *equation* may often be greatly simplified by a judicious change of the axes to which it is referred. *E.g.*, the line L_1L, when referred to the axes OX and OY, has the equation

$$y = \tan\theta \cdot x + b,$$

but when referred to the axes $O'X'$ and $O'Y'$, the former of which is parallel to the given line, its equation is $y = c$.

For these, and other reasons, in the study of curves and surfaces by the methods of analytic geometry, it will often be found advantageous to transform the equations from one set of axes to another.

It will be found that the coördinates of a point with reference to any given axes, are always connected by simple formulas with the coördinates of the same point when it is referred to any other axes. These relations or formulas for the various changes of axes are derived in the next few articles.

I. CARTESIAN COÖRDINATES ONLY

71. Change of origin, new axes parallel respectively to the original axes. Let OX and OY be the original axes, $O'X'$ and $O'Y'$ the new axes, and let the coördinates of the new origin when referred to the original axes be h and k, *i.e.*, $O' \equiv (h, k)$, where $h = OA$ and $k = AO'$. Also let P, any point of the plane, have the coördinates x and y when it is referred to the axes OX and OY, and x' and y' when it is referred to the axes $O'X'$ and $O'Y'$.

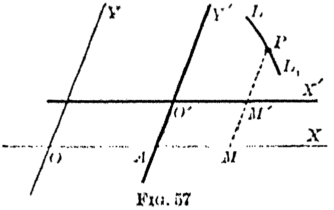


FIG. 57

Draw $MM'P$ parallel to the y-axis; then

$$OM = OA + AM = OA + O'M',$$

i.e., and similarly,

$$\left.\begin{aligned} x &= x' + h, \\ y &= y' + k, \end{aligned}\right\} \quad . \quad . \quad . \qquad [23]$$

which are the equations (or formulas) of transformation from any given axes to new axes which are respectively parallel to the original ones, the new origin being the point $O' \equiv (h, k)$. These formulas, moreover, are independent of the angle between the axes.

As a simple illustration of the usefulness of such a change of axes, suppose the equation

$$x^2 - 2hx + y^2 - 2ky = a^2 - h^2 - k^2 \quad . \quad . \quad (1)$$

given, in which x and y are coördinates referred to the axes OX and OY.

Now let $P \equiv (x, y)$ be any point on the locus L_1L of this equation, and let (x', y') be the coördinates of the same point P when it is referred to the axes $O'X'$ and $O'Y'$; then

$$x = x' + h \quad \text{and} \quad y = y' + k.$$

Substituting these values in the given equation for the x and y there involved, an equation in x' and y' is obtained which is satisfied by the coördinates of every point on L_1L, *i.e.*, it is the equation of the same locus. The substitution gives:

$$(x' + h)^2 - 2h(x' + h) + (y' + k)^2 - 2k(y' + k) = a^2 - h^2 - k^2,$$

which reduces to

$$x'^2 + y'^2 = a^2;$$

a much simpler equation than (1), but representing the same locus, merely referred to other axes.

EXERCISES

1. What is the equation for the locus of $3x - 2y = 6$, if the origin be changed to the point $(4, 3)$, — directions of axes unchanged?

2. What does the equation $x^2 + y^2 - 4x - 6y = 18$ become if the origin be changed to the point $(2, 3)$, — directions of axes unchanged?

3. What does the equation $y^2 - 2x^2 - 2y + 6x - 3 = 0$ become when the origin is removed to $(\frac{3}{2}, 1)$, — directions of axes unchanged?

4. Find the equation for the straight line $y = 3x + 1$ when the origin is removed to the point $(1, 4)$, — directions of axes unchanged.

5. Construct appropriate figures for exercises 1 and 4.

72. Transformation from one system of rectangular axes to another system, also rectangular, and having the same origin: change of direction of axes.

Let OX and OY be a given pair of rectangular axes, and let OX' and OY' be a second pair, with $\angle XOX' = \theta$, the angle through which the first pair of axes must be turned to come into coincidence with the second. Also let P, any point in the plane, have the coördinates x and y when it is referred to the first pair of axes, and x' and y' when referred to the second pair. The problem now is to express x and y in terms of x', y', and θ. Draw the ordinates MP, $M'P$, and QM', and draw $M'R$ parallel to the x-axis; then

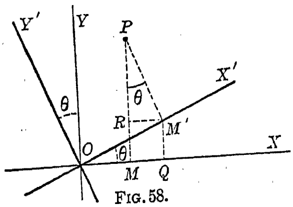


FIG. 58.

$$OM = OQ + QM = OM' \cos\theta - M'P \sin\theta,$$

i.e.,
$$\left.\begin{aligned} x &= x' \cos\theta - y' \sin\theta, \\ y &= x' \sin\theta + y \cos\theta, \end{aligned}\right\} \quad . \quad . \quad . \quad [24]$$
and similarly,

which are the required formulas of transformation from one pair of rectangular axes to another, having the same origin but making an angle θ with the first pair.

NOTE 1. These formulas are more easily obtained, — in fact, they can be read directly from the figure, — if one recalls Art. 17, and considers that the projection of OP equals the projection of OM' + the projection of $M'P$, upon OX and OY in turn.

NOTE 2. The reader will observe that a combination of the transformation of Art. 71 with that of Art. 72 will transform from one pair of rectangular axes to *any* other pair of rectangular axes.

EXERCISES

Turn the axes through an angle of 45°, and find the new equations for the following loci:

1. $x^2 + y^2 = 16$; **2.** $x^2 - y^2 = 16$; **3.** $y = x - 1$;

4. $17x^2 - 16xy + 17y^2 = 225$.

5. If the axes are turned through the angle $\tan^{-1} 2$, what does the equation $4xy - 3x^2 = a^2$ become?

73. Transformation from rectangular to oblique axes, origin unchanged. Let OX and OY be a given pair of rectangular axes, let OX' and OY' be the new axes making an angle ω with each other, and let the angles XOX' and XOY' be denoted by θ and ϕ, respectively. Also let P, any point in the plane, have the coördinates x and y when referred to the first pair of axes, and x' and y' when referred to the second pair.

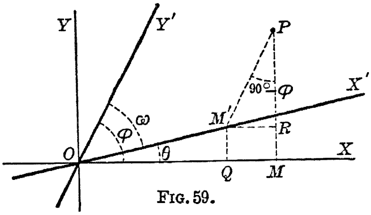


FIG. 59.

Draw the ordinates MP, $M'P$, and QM', also draw $M'R$ parallel to the x-axis.

Then $OM = OQ + QM = OM' \cos\theta + M'P \sin(90 - \phi)$;

$$\begin{aligned} &i.e., & x &= x' \cos\theta + y' \cos\phi, \\ &\text{and similarly,} & y &= x' \sin\theta + y' \sin\phi, \end{aligned}\Bigg\}^* \quad . \; . \; . \; . \; . \quad [25]$$

which are the required formulas of transformation from rectangular to oblique axes having the same origin.

If $\omega = 90°$, and consequently $\phi = 90° + \theta$, then formulas [25] reduce to [24], and Art. 73, therefore, includes Art. 72 as a special case.

By first solving for x' and y', formulas [25] may also be employed to transform from oblique to rectangular axes.

---

* See NOTE 1, Art. 72.

EXERCISES

1. Given the equation $9x^2 - 16y^2 = 144$ referred to rectangular axes; what does this equation become if transformed to new axes such that the new x-axis makes the angle $\tan^{-1}(-\frac{3}{4})$, and the new y-axis the angle $\tan^{-1}(\frac{3}{4})$, with the old x-axis, — origin unchanged?

2. If the old and new x-axes coincide, and the new axes are rectangular while the old axes are inclined at an angle of 60°, what are the equations of transformation from the old axes to the new? From the new axes to the old? Origin unchanged in each case.

3. If the first two of the three sides of a triangle whose equations are $2y + x + 1 = 0$, $3y - x - 1 = 0$, and $2x + 3y = 1$, are chosen as new axes, find the new equations of the sides.

74. Transformation from one set of oblique axes to another, origin unchanged. Let OX and OY be a given pair of axes, OX' and OY' the new axes, and let the angles XOY, $X'OY'$, XOX', and XOY' be denoted by ω, ω', θ, and ϕ, respectively. Also let P, any point in the plane, have the coördinates x and y when referred to the first pair of axes, and x' and y' when referred to the second pair.

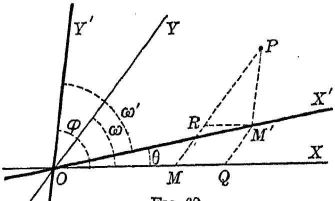


FIG. 60.

Draw $M'P$ parallel to OY', MP and QM' parallel to OY, and $M'R$ parallel to OX.

Then, from the triangle OQM',

$$OQ = x' \frac{\sin(\omega - \theta)}{\sin \omega} \text{ and } QM' = x' \frac{\sin \theta}{\sin \omega},$$

and from the triangle $RM'P$,

$$RM' = y' \frac{\sin(\phi - \omega)}{\sin \omega} \text{ and } RP = y' \frac{\sin \phi}{\sin \omega}.$$

But $OM = OQ - RM'$, and $MP = QM' + RP$;

$$\left.\begin{aligned}\therefore\ x &= x'\frac{\sin(\omega-\theta)}{\sin\omega}+y'\frac{\sin(\omega-\phi)}{\sin\omega},\\ \text{and } y &= x'\frac{\sin\theta}{\sin\omega}+y'\frac{\sin\phi}{\sin\omega},\end{aligned}\right\}^{*}\quad\ldots\quad[26]$$

which are the required formulas of transformation from one pair of oblique axes to another having the same origin.

NOTE. If it is desired to change the *origin*, and also the direction of the axes, the necessary formulas may be obtained by combining Art. 71 with Art. 72, Art. 73, or Art. 74, depending upon the given and required axes.

EXERCISES

1. Show, by specializing some of the angles ω, ω', θ, and ϕ in Art. 74, that formulas [26] include both [25] and [24] as special cases.

2. The equation of a certain locus, when referred to a pair of axes that are inclined to each other at an angle of 60°, is $7x^2-2xy+4y^2=5$; what will this equation become if the axes are each turned through an angle of 30°? What if the x-axis is turned through the angle $-30°$ while the y-axis is turned through $+30°$?

75. The degree of an equation in Cartesian coördinates is not changed by transformation to other axes. Every formula of transformation obtained ([23] to [26]) has replaced x and y, respectively, by expressions of the first degree in the new coördinates x', y'. Therefore any one of these transformations replaces the terms containing x and y by expressions of the same degree, and so cannot *raise* the degree of the given equation. Neither can any one of these transformations *lower* the degree of the given equation; for if it did, then a transformation back to the original axes (which must give again the original equation) would *raise* the degree, which has just been shown to be impossible; hence all these transformations leave the degree of an equation unchanged.

---

* These formulas can also be read directly from Fig. 60 by first projecting OM and then the broken line $OM'PM$ upon a line perpendicular to OY; and afterwards projecting MP and also the broken line $MOM'P$ upon a perpendicular to OX. The results being equated in each case, and divided by $\sin\omega$, give [26].

II. POLAR COÖRDINATES

76. Transformations between polar and rectangular systems. (1) *Transformation from a rectangular to a polar system, and vice versa, the origin and x-axis coinciding respectively with the pole and the initial line.* Let OX and OY be a given set of rectangular axes, and let OX and O be the initial line and pole for the system of polar coördinates. Also let P, any point in the plane, have the coördinates x and y when referred to the rectangular axes, and ρ and θ in the polar system (Fig. 61), then

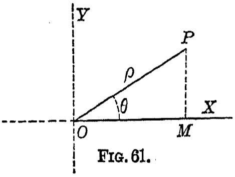


FIG. 61.

$$OM = OP \cos XOP;$$

i.e., similarly,

$$\left.\begin{aligned} x &= \rho \cos \theta; \\ y &= \rho \sin \theta. \end{aligned}\right\} \quad . \quad . \quad . \quad [27]$$

These are the required formulas of transformation when, *but only when*, the rectangular and polar axes are related as above described.

Conversely, from formulas [27], or directly from Fig. 61,

$$\rho = \sqrt{x^2 + y^2},\ \cos\theta = \frac{x}{\sqrt{x^2+y^2}},\ \text{and } \sin\theta = \frac{y}{\sqrt{x^2+y^2}}, \quad [28]$$

which are the required formulas of transformation from polar to rectangular axes, under the above conditions.

(2) *Same as* (1) *except that the initial line OR makes an angle* α *with the x-axis.* It is at once evident that the formulas of transformation for this case are:

$$\left.\begin{aligned} x &= \rho\cos(\theta+\alpha), \\ \text{and}\quad y &= \rho\sin(\theta+\alpha). \end{aligned}\right\}\;\ldots\;[29]$$

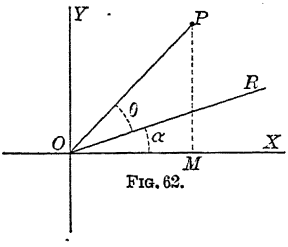


FIG. 62.

The converse formulas for this case are:

$$\rho = \sqrt{x^2+y^2}$$

$$\text{and}\quad \theta = \cos^{-1}\left(\frac{x}{\sqrt{x^2+y^2}}\right) - \alpha = \sin^{-1}\left(\frac{y}{\sqrt{x^2+y^2}}\right) - \alpha. \quad [30]$$

(3) *Transformation from any Cartesian system to any polar system.* Transform first to rectangular axes whose origin is the proposed pole; this is accomplished by Arts. 71 and 73. Then by formula [27] or [29] transform from the rectangular Cartesian to the polar coördinates.

EXERCISES

Change the following to the corresponding polar equations; draw a figure showing the two related systems of axes in each case. Take the pole at the origin, the polar axis coincident with the axis of x, in exercises 1 to 4.

1. $x^2 + y^2 = a^2$.

2. $y^2 - x + 2ay = 0$.

3. $x^2 + y^2 = 9(x^2 - y^2)$.

4. $y = x\tan\alpha$.

5. $x - \sqrt{3}\,y + 2 = 0$, taking pole at origin, polar axis making the angle 60° with the x-axis.

6. $x^2 - y^2 - 4x - 6y - 54 = 0$, taking the pole at the point $(2, {}^{-}3)$, and the polar axis parallel to the x-axis.

Change the following to corresponding rectangular equations. Take the origin at the pole and the x-axis coincident with the polar axis.

7. $\rho = a$.

8. $\rho^2\cos 2\theta = a^2$.

9. $\rho^2\sin 2\theta = 10$.

10. $\rho^2 = a^2\sin 2\theta$.

SUGGESTION. In Ex. 10 multiply by ρ^2 and substitute $2\sin\theta\cos\theta$ for $\sin 2\theta$; the equation then becomes $\rho^4 = 2a^2\rho^2\sin\theta\cos\theta$.

11. $\rho = k\cos\theta$. **12.** $\theta = 3\tan^{-1}2$. **13.** $\rho^{\frac{1}{2}}\cos\frac{\theta}{2} = k^{\frac{1}{2}}$.

EXAMPLES ON CHAPTER VI

1. Find the equation of the locus of $2xy - 7x + 4y = 0$ referred to parallel axes through the point $(^-2, \frac{7}{2})$.

2. Transform the equation $x^2 - 4xy + 4y^2 - 6x + 12y = 0$ to new rectangular axes making an angle $\tan^{-1}\frac{1}{2}$ with the given axes.

3. Transform $y^2 - xy - 5x + 5y = 0$ to parallel axes through the point $(^-5, {}^-5)$. Draw an appropriate figure.

4. Transform the equation of example 3 to axes bisecting the angles between the old axes. Trace the locus.

5. To what point must the origin be moved (the new axes being parallel to the old) in order that the new equation of the locus

$$2x^2 - 5xy - 3y^2 - 2x + 13y - 12 = 0$$

shall have no terms of first degree?

SOLUTION. Let the new origin be (h, k); then $x = x' + h$, $y = y' + k$, and the new equation is

$$2(x'+h)^2 - 5(x'+h)(y'+k) - 3(y'+k)^2 - 2(x'+h) + 13(y'+k) - 12 = 0,$$

i.e., $$2x'^2 - 5x'y' - 3y'^2 + (4h - 5k - 2)x' - (5h + 6k - 13)y' + 2h^2 - 5hk - 3k^2 - 2h + 13k - 12 = 0;$$

but it is required that the coefficients of x' and y' shall be 0; *i.e.*, h and k are to be determined so that

$$4h - 5k - 2 = 0,$$

and $$5h + 6k - 13 = 0;$$

hence $$h = \tfrac{11}{7} \text{ and } k = \tfrac{6}{7}.$$

Therefore the new origin must be at the point $(\frac{11}{7}, \frac{6}{7})$, and the new equation is

$$2x'^2 - 5x'y' - 3y'^2 - 8 = 0.$$

6. The new axes being parallel to the old, determine the new origin so that the new equation of the locus

$$x^2 - 3xy + y^2 + 10x - 10y + 21 = 0$$

shall have no terms of first degree.

7. Transform the equations $x + y - 3 = 0$ and $2x - 3y + 4 = 0$ to parallel axes having the point of intersection of these lines as origin.

8. Transform the equation $\frac{x}{4} + \frac{y}{6} = 1$ to new rectangular axes through the point (2, 3), and making the angle $\tan^{-1}(-\frac{3}{2})$ with the old axes.

9. Through what angle must the axes be turned that the new equation of the line $6x + 4y - 24 = 0$ shall have no y-term? Show this geometrically, from a figure.

10. Through what angle must the axes be turned in order that the new equation of the line $6x + 4y = 24$ shall have no x-term? Show analytically (cf. also examples 8 and 9).

Solution. Let θ be the required angle; then the equations of transformation are

$$x = x' \cos\theta - y' \sin\theta \text{ and } y = x' \sin\theta + y' \cos\theta;$$

and the new equation is

$$(6\cos\theta + 4\sin\theta)x - (6\sin\theta - 4\cos\theta)y = 24;$$

but it is required that the coefficient of x be 0,

$$\therefore \qquad 6\cos\theta + 4\sin\theta = 0, \text{ i.e., } \tan\theta = -\tfrac{6}{4};$$

whence $$\theta = \tan^{-1}(-\tfrac{3}{2}),$$

and the equation becomes

$$(6\sin\theta - 4\cos\theta)y + 24 = 0,$$

which reduces to $$\frac{10}{\sqrt{13}}y + 24 = 0,$$

i.e., to $$5y + 12\sqrt{13} = 0.$$

11. Through what angle must the axes be turned to remove the x-term from the equation of the locus $Ax + By + C = 0$? to remove the y-term?

12. Show that to remove the xy-term from the equation of the locus, $2x^2 - 5xy - 3y^2 = 8$ (cf. Ex. 5), the axes must be turned through the angle $\theta = 67° 30'$, *i.e.*, so that $\tan 2\theta = -1$. What is the new equation?

13. Through what angle must a pair of rectangular axes be turned that the new x-axis may pass through the point $(-2, -5)$?

14. What point must be the new origin, the direction of axes being unchanged, in order that the new equation of the line $Ax + By + C = 0$ shall have no constant term?

15. To what point, as origin of a pair of parallel axes, must a transformation of axes be made in order that the new equation of the locus, $xy - y^2 - x + y = 0$, shall have no terms of first degree? Construct the locus.

16. Find the new origin, the direction of axes remaining unchanged, so that the equation of the locus, $x^2 + xy - 3x - y + 2 = 0$, shall have no constant term. Construct the figure.

17. Transform the equation $4x^2 + 2\sqrt{3}xy + 2y^2 = 1$ to new rectangular axes making an angle of 30° with the given axes,—origin unchanged.

18. Transform $y^2 = 8x$ to new rectangular axes having the point (18, 12) as origin, and making an angle $\cot^{-1} 3$ with the old.

19. Transform to rectangular coördinates, the pole and initial line being coincident with the origin and x-axis, respectively:

$$(\alpha)\ \rho^2 = a^2 \cos 2\theta, \quad (\beta)\ \rho^2 \cos 2\theta = a^2, \quad (\gamma)\ \rho = k \sin 2\theta.$$

Transform to polar coördinates, the x-axis and initial line being coincident:

20. $(x^2 + y^2)^2 = k^2(x^2 - y^2)$, the pole being at the point (0, 0);

21. $x^2 + y^2 = 7ax$, pole being at the point (0, 0);

22. $x^2 + y^2 = 16x$, the pole being at the point (8, 0).

23. Transform the equation $y^2 + 4ay \cot 30° - 4ax = 0$ to an oblique system of coördinates, with the same origin and x-axis, but the new y-axis at an angle of 30° with the old x-axis.

24. Transform the equation $\frac{x^2}{16} + \frac{y^2}{9} = 1$, to new axes, making the positive angles $\tan^{-1} \frac{3}{4}$ and $\tan^{-1}(-\frac{3}{4})$, respectively, with the old x-axis, the origin being unchanged.

25. Transform the equation

$$3x^2 + 10\sqrt{3}xy - 7y^2 = (18 - 30\sqrt{3})x + (42 + 30\sqrt{3})y + (42 + 90\sqrt{3})$$

to the new origin $(3, -3)$, with new axes making an angle of 30° with the old.

26. Transform the equation $3x^2 + 8xy - 3y^2 = 0$ to the two straight lines which it represents, as new axes.

27. Transform $\frac{x^2}{25} - \frac{y^2}{9} = 1$ to the straight lines $\frac{x^2}{25} - \frac{y^2}{9} = 0$, as new axes.

28. Transform to polar coördinates the equation $y^2(2a - x) = x^3$.

29. Transform to rectangular coördinates the equation

$$\rho = a(\cos 2\theta + \sin 2\theta).$$

30. Prove the formula for the distance in polar coördinates [1] by transformation of the corresponding formula [2] in rectangular coördinates.

31. Transform the equation $x \cos \alpha + y \sin \alpha = p$ to polar coördinates.

CHAPTER VII

THE CIRCLE

Special Equation of the Second Degree

$$Ax^2 + Ay^2 + 2Gx + 2Fy + C = 0$$

77. It must be kept clearly in mind that one of the chief aims of an elementary course in Analytic Geometry is to teach a new *method* for the study of geometric properties of curves and surfaces. Power and facility in the use of such a new method are best acquired by applying it first to those loci whose properties are already best understood. Accordingly, the straight line having already been studied in Chapter V, the circle will next be examined.

It will appear later that the circle is only a special case of the conic sections already referred to in Art. 48, and might, therefore, be advantageously studied *after* the general properties of those curves had been examined; the present order is adopted, however, because the student is already familiar with the chief properties of the circle.

In solving the exercises of this chapter the student should use the analytic methods, even when purely geometric methods might suffice, — he is learning to use a new instrument of investigation, and is not merely studying the properties of the circle.

78. The circle: its definition, and equation. The circle may be defined as the path traced by a point which moves in such a way as to be always at a constant distance from a given fixed point. This fixed point is the **center**, and the constant distance is the **radius**, of the circle.

To derive the equation from this definition, let $C \equiv (h, k)$ be the center, r the radius, and $P \equiv (x, y)$ any point on the curve. Also draw the ordinates M_1C and MP, and the line CR parallel to the x-axis; then

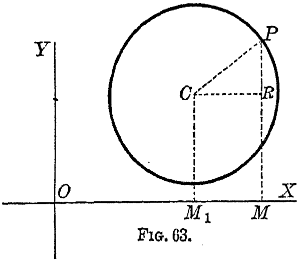


FIG. 63.

$CP = r$; [geometric equation]

but (Art. 26),

$$CP = \sqrt{(x-h)^2 + (y-k)^2},$$

hence $\sqrt{(x-h)^2 + (y-k)^2} = r$;

i.e., $$(x-h)^2 + (y-k)^2 = r^2,\text{*} \quad . \quad . \quad . \quad [31]$$

which is the equation of the circle whose radius is r, and whose center has the coördinates h and k.

With given fixed axes, equation [31] may, by rightly choosing h, k, and r, represent any circle whatever; it is, therefore, called the *general equation of the circle.* Of its special forms one is, because of its very frequent application, particularly important; this form is the one for which the center coincides with the origin: in that case $h = k = 0$, and equation [31] becomes

$$x^2 + y^2 = r^2.\dagger \quad . \quad . \quad . \quad [32]$$

---

* Equation [31] may be written in the form

$$(x-h)^2 + (y-k)^2 - r^2 = 0;$$

the first member then becomes positive if the coördinates of any point *outside* of the circle are substituted for x and y, it becomes negative for a point *inside* of the circle, and zero for a point *on* the circle. Hence the circle may be regarded as the boundary which separates that part of the plane for which the function $(x-h)^2 + (y-k)^2 - r^2$ is positive from the part for which this function is negative. The inside of the circle may therefore be called its *negative* side, while the outside is its *positive* side (cf. foot-note, Art. 43).

† If one is unrestricted in his choice of axes, then an equation of the form of [32] may represent any circle whatever, — the axes need merely be chosen perpendicular to each other and through its center; — equation [31] is more general in that, the rectangular axes being determined by other considerations, it may still represent any circle whatever.

EXERCISES

First construct the circle, then find its equation, being given

1. the center $(5, {}^-3)$, the radius 4;
2. the center $(0, 2)$, the radius $\frac{3}{2}$;
3. the center $(3, {}^-3)$, the radius 3;
4. the center $(0, 0)$, the radius 5;
5. the center $({}^-4, 0)$, the radius 1.

6. How are circles related for which h and k are the same, while r is different for each? for which h and r are the same, while k differs for each?

7. What form does the equation of the circle assume when the center is on the x-axis and the origin on the circumference? when the circle touches each axis and has its center in quadrant II?

79. In rectangular coördinates every equation of the form $x^2 + y^2 + 2Gx + 2Fy + C = 0$ represents a circle. The equations of the circles already obtained (equations [31] and [32], as well as the answers to examples 1 to 5 and 7) are all of the form

$$x^2 + y^2 + 2Gx + 2Fy + C = 0; \quad . \quad . \quad . \quad (1)$$

it will now be shown that, for all values of G, F, and C, for which the locus of equation (1) is real, this equation represents a circle.

To prove this it is only necessary to complete the square in the x-terms and in the y-terms, by adding $G^2 + F^2$ to each member of equation (1), and then transpose C to the second member. Equation (1) may then be written in the form

$$\begin{aligned}(x + G)^2 + (y + F)^2 &= G^2 + F^2 - C \\ &= (\sqrt{G^2 + F^2 - C})^2 \quad . \quad . \quad (2)\end{aligned}$$

which is (cf. equation [31]) the equation of a circle whose center is the point $(-G, -F)$, and whose radius is

$$\sqrt{G^2 + F^2 - C}.$$

NOTE 1. This circle is *real* only if $G^2 + F^2 - C > 0$; for, if

$$G^2+F^2 - C<0,$$

its square root is imaginary, and no real values of x and y can then satisfy equation (2); while if $G^2 + F^2 - C = 0$, then equation (2) reduces to

$$(x + G)^2 + (y + F)^2 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which may be called the equation of a "point circle," since the coördinates of but one real point, viz. $(-G, -F)$, will satisfy equation (3). If, however, $G^2 + F^2 - C > 0$, then equation (1) represents a real circle for all values of G, F, and C, subject to this single limitation.

NOTE 2. Every equation of the form $Ax^2 + Ay^2 + 2\,Gx + 2\,Fy + C = 0$ represents a circle, for, by Art. 38, this equation has the same locus as has $x^2 + y^2 + 2\frac{G}{A}x + 2\frac{F}{A}y + \frac{C}{A} = 0$, and this last equation is of the form of equation (1).

Hence, interpreted in rectangular coördinates, every equation of the second degree from which the term in xy is absent, and in which the coefficient of x^2 equals that of y^2, represents a circle.

80. Equation of a circle through three given points. By means of equation [31], or of the equation

$$x^2 + y^2 + 2\,Gx + 2\,Fy + C = 0, \quad . \quad . \quad . \quad (1)$$

which has been shown in Art. 79 to be its equivalent, the problem of finding the equation of a circle which shall pass through any three given points not lying on a straight line can be solved; *i.e.*, the constants h, k, and r, or G, F, and C, may be so determined that the circle shall pass through the three given points.

To illustrate: let the given points be (1, 1), (2, $^-1$), and (3, 2), and let $x^2 + y^2 + 2\,Gx + 2\,Fy + C = 0$ be the equation of the circle that passes through these points; to find the values of the constants G, F, and C. Since the point (1, 1) is on this circle, therefore (cf. Art. 35),

$$1 + 1 + 2\,G + 2\,F + C = 0\,;$$

similarly, $4+1+4G-2F+C=0,$

and $9+4+6G+4F+C=0.$

These equations give: $G=-\frac{5}{2}$, $F=-\frac{1}{2}$, and $C=4$. Substituting these values, the equation of the required circle becomes

$$x^2+y^2-5x-y+4=0;$$

its center is at the point $(\frac{5}{2}, \frac{1}{2})$, while its radius is $\frac{1}{2}\sqrt{10}$.

NOTE. The fact that the most general equation of the circle contains three parameters (h, k, and r, or G, F, and C, above) corresponds to the property that a circle is uniquely determined by three of its points.

EXERCISES

Find the radii, and the coördinates of the centers, of the following circles; also, draw the circles.

1. $x^2+y^2-4x-8y-41=0.$

2. $3x^2+3y^2-5x-7y+1=0.$

3. $x^2+y^2=3(x+3).$

4. $2(x^2+y^2)=7y.$

5. $ax^2+ay^2=bx+cy.$

6. $(x+y)^2+(x-y)^2=8a^2.$

7. What loci are represented by the equations

$$(x-h)^2+(y-k)^2=0,$$

and

$$x^2+y^2-2x+6y+38=0?$$

Find the equation of the circle through the points:

8. $(1, 2)$, $(3, {}^-4)$, and $(5, {}^-6)$;

9. $(0, 0)$, (a, b), and (b, a);

10. $({}^-6, {}^-1)$, $(0, 1)$, and $(1, 0)$;

11. $(10, 2)$, $(3, 3)$, and having the radius 2.

12. Find the equation of the circle which has the line joining the points $(3, 4)$ and $({}^-1, 2)$ for a diameter.

13. Find the equation of the circle which touches each axis, and passes through the point $({}^-2, 3)$.

14. A circle has its center on the line $3x+4y=7$, and touches the two lines $x+y=3$ and $x-y=3$; find its equation, radius, and center; also draw the circle.

SECANTS, TANGENTS, AND NORMALS

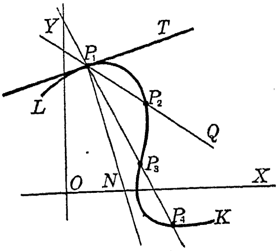


Fig. 64.

81. Definitions of secants, tangents, and normals. A straight line will, in general, intersect any given curve in two or more distinct points; it is then called a **secant** line to the curve. Let P_1 and P_2 be two successive points of intersection of a secant line P_1P_2Q with a given curve $LP_1P_2 \dots K$; if this secant line be rotated about the point P_1 so that P_2 approaches P_1 along the curve, the limiting position P_1T which the secant approaches, as P_2 approaches coincidence with P_1, is called a **tangent** to the curve at that point. This conception of the tangent leads to a method, of extensive application, for deriving its equation, — the so-called "secant method." *

Since the points of intersection of a line and a curve are found (Art. 39) by considering their equations as simultaneous, and solving for x and y, it follows that, if the line is tangent to the curve, the abscissas of two points of intersection, as well as their ordinates, are equal. Therefore, if the line is a tangent, the equation obtained by eliminating x or y between the equation of the line and that of the curve must have a pair of equal roots.

If the given curve is of the second degree, then the equation resulting from this elimination is of the second degree, and the test for equal roots is well known (Art. 9); but if the given equation is of a degree higher than the second, other methods must in general be used.

A straight line drawn perpendicular to a tangent and through the point of tangency is called a **normal** line to the curve at that point. Thus, in Fig. 64, P_1P_2, P_1P_3 are secants, P_1T is a tangent, and P_1N a normal to the curve at P_1.

---

* For illustration, see Art. 84.

82. Tangents: Illustrative examples.

(1) To find the equation of that tangent to the circle $x^2 + y^2 = 5$ which makes an angle of 45° with the x-axis. Since this line makes an angle of 45° with the x-axis its equation is $y = x + b$, where b is to be determined so that this line shall *touch* the circle.

Clearly, from the figure, there are two values of b (OB_1 and OB_2) for which this line will be tangent to the circle. According to Art. 81, these values of b are those which make the two points of intersection of the line and the circle become coincident.

Considering the equations $x^2 + y^2 = 5$ and $y = x + b$ simultaneous, and eliminating y, the resulting equation in x is $x^2 + (x + b)^2 = 5$, *i.e.*, $2x^2 + 2bx + b^2 - 5 = 0$. The roots of this equation will become equal, *i.e.*, the abscissas of the points of intersection will become equal (Art. 9), if $b^2 - 2(b^2 - 5) \doteq 0$, *i.e.*, if $b \doteq \pm\sqrt{10}$.

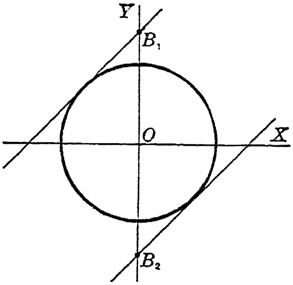


Fig. 65.

The equations of the two required tangent lines are, therefore,

$$y = x + \sqrt{10}, \text{ and } y = x - \sqrt{10}.$$

(2) To find the equations of those tangents to the circle $x^2 + y^2 = 6y$ that are parallel to the line $x + 2y + 11 = 0$.

The equation of a line parallel to $x + 2y + 11 = 0$ is $x + 2y + k = 0$, where k is an arbitrary constant (Art. 62), and this line will become tangent to the circle, if the value of the constant k be so chosen that the two points in which the line meets the circle shall become coincident.

Considering the equations $x^2 + y^2 = 6y$ and $x + 2y + k = 0$ simultaneous, and eliminating x, the resulting equation in y is

$$(-k - 2y)^2 + y^2 = 6y, \text{ i.e., } 5y^2 + (4k - 6)y + k^2 = 0.$$

The two values of y will become equal if (Art. 9)

$$(4k - 6)^2 - 20k^2 \doteq 0, \text{ i.e., if } k^2 + 15k - 9 \doteq 0,$$

i.e., if $$k \doteq -6 \pm 3\sqrt{5},$$

and the two required tangent lines are:

$$x + 2y - 6 + 3\sqrt{5} = 0, \text{ and } x + 2y - 6 - 3\sqrt{5} = 0.$$

EXERCISES

Find the equations of the tangents:

1. to the circle $x^2 + y^2 = 4$, parallel to the line $x + 2y + 3 = 0$;

2. to the circle $3(x^2 + y^2) = 4y$, perpendicular to the line $x + y = 7$;

3. to the circle $x^2 + y^2 + 10x - 6y - 2 = 0$, parallel to the line $y = 2x - 7$;

4. to the circle $x^2 + y^2 = r^2$, and forming with the axes a triangle whose area is r^2.

5. Show that the line $y = x + c\sqrt{2}$ is, for all values of c, tangent to circle $x^2 + y^2 = c^2$; and find, in terms of c, the point of contact.

6. Prove that the circle $x^2 + y^2 + 2x + 2y + 1 = 0$ touches both coördinate axes; and find the points of contact.

7. For what values of c will the line $3x - 4y + c = 0$ touch the circle $x^2 + y^2 - 8x + 12y - 44 = 0$?

8. For what value of r will the circle $x^2 + y^2 = r^2$ touch the line $y = 3x - 5$?

9. Prove that the line $ax = b(y - b)$ touches the circle $x(x - a) + y(y - b) = 0$; and find the point of contact.

10. Three tangents are drawn to the circle $x^2 + y^2 = 9$; one of them is parallel to the x-axis, and together they form an equilateral triangle. Find their equations, and the area of the triangle.

83. Equation of tangent to the circle $x^2 + y^2 = r^2$ in terms of its slope. The equation of the tangent to a given circle, in terms of its slope, is found in precisely the same way as that followed in solving (1) of Art. 82. Let m_1 be the given slope of the tangent, then the equation of the tangent is of the form

$$y = m_1x + b, \quad . \quad . \quad . \quad (1)$$

wherein b is a constant which must be so determined that line (1) shall intersect the circle

$$x^2 + y^2 = r^2 \quad . \quad . \quad . \quad (2)$$

in two coincident points.

Eliminating y between equations (1) and (2) gives

$$x^2 + (m_1x + b)^2 = r^2,$$

i.e., $$x^2(1 + m_1^2) + 2\,bm_1x + b^2 - r^2 = 0\,;$$

and the two values of x obtained from this equation will become equal (Art. 9) if

$$(m_1b)^2 - (1 + m_1^2)(b^2 - r^2) \doteq 0,$$

i.e., if $$b \doteq \pm r\sqrt{1 + m_1^2}.$$

Substituting this value of b in equation (1), it becomes

$$\boldsymbol{y = m_1x \pm r\sqrt{1 + m_1^2}},^* \quad . \quad . \quad . \qquad [33]$$

which is then, for all values of m_1, tangent to the circle (2).

This equation [33] enables one to write down immediately the equation of a tangent, of given slope, to a circle *whose center is at the origin.*

E.g., to find the equation of the tangent whose slope $m_1 = 1 = \tan 45°$, to the circle $x^2 + y^2 = 5$, it is only necessary to substitute 1 for m_1 and $\sqrt{5}$ for r in equation [33]. This gives as the required equation $y = x \pm \sqrt{10}$ [cf. (1) Art. 82].

NOTE 1. If the center of the given circle is not at the origin, *i.e.*, if its equation is of the form $x^2 + y^2 + 2\,Gx + 2\,Fy + C = 0$, instead of $x^2 + y^2 = r^2$, then the same reasoning as that employed above would lead to

$$y + F = m_1(x + G) \pm \sqrt{G^2 + F^2 - C}\cdot\sqrt{1 + m_1^2}, \quad . \quad . \qquad [34]$$

as the equation of the required tangent.

This equation might have been obtained also by first transforming the equation $x^2 + y^2 + 2\,Gx + 2\,Fy + C = 0$ to parallel axes through the point $(-G, -F)$; this would have given $x'^2 + y'^2 = G^2 + F^2 - C = r^2$ as the equation of the *same* circle, but now referred to axes through its center. Referred to these new axes $y' = m_1x' \pm r\sqrt{1 + m_1^2}$ (see eq. [33]) is, for all values of m_1, tangent to this circle; transforming this last equation back to the original axes, *i.e.*, substituting for x', y', and r their equals, viz., $x + G$, $y + F$, and $\sqrt{G^2 + F^2 - C}$, it becomes

$$y + F = m_1(x + G) \pm \sqrt{G^2 + F^2 - C}\cdot\sqrt{1 + m_1^2}$$

---

* This equation is sometimes spoken of as the *magical* equation of the tangent.

as before, which is, for all values of m_1, tangent to the circle whose center is at the point $(-G, -F)$ and whose radius is $\sqrt{G^2 + F^2 - C}$.

Note 2. Because of its frequent occurrence, it is useful to memorize equation [33]. On the other hand, it is not recommended that equation [34] be memorized, but it should be carefully worked out by the student. Instead of employing either of these formulas, however, the student may always attack the problems directly, as was done in Art. 82.

EXERCISES

Find the equations of the lines which are tangent:

1. to the circle $x^2 + y^2 = 16$, and whose slope is 3;

2. to the circle $x^2 + y^2 = 1$, and which are parallel to the line $x + 2y - 7 = 0$ (cf. Ex. 1, Art. 82);

3. to the circle $x^2 + y^2 = 9$, and which make an angle of 60° with the x axis; with the y axis;

4. to the circle $x^2 + y^2 = 25$, and which are perpendicular to the line joining the points $(-3, 7)$ and $(7, 5)$;

5. to the circle $x^2 + y^2 - 2x + 2y = 1$, and whose slope is -1.

84. Equation of tangent to the circle in terms of the coördinates of the point of contact: the secant method.

(*a*) *Center of the circle at the origin.* Let $P_1 \equiv (x_1, y_1)$ be the point of tangency, on the given circle

$$x^2 + y^2 = r^2. \quad . \quad . \quad . \quad (1)$$

Through P_1 draw a secant line LM, and let $P_2 \equiv (x_2, y_2)$ be its other point of intersection with the circle. If the point P_2 moves *along the circle* until it comes into coincidence with P_1, the limiting position of the secant LM is the tangent P_1T. (Art. 81.)

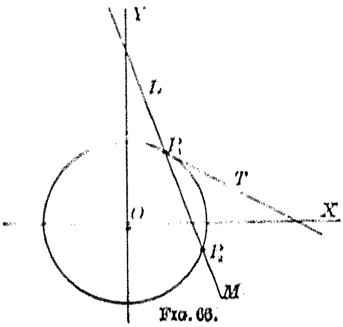


Fig. 66.

The equation of the line LM is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1). \quad . \quad . \quad . \quad (2)$$

If now P_2 approaches P_1 until

$x_2=x_1$ and $y_2=y_1$, equation (2) takes the indeterminate form

$$y-y_1=\frac{0}{0}(x-x_1). \quad . \quad . \quad . \quad (3)$$

This indeterminateness arises because account has not yet been taken of the path (or direction) by which P_2 shall approach P_1, and it disappears immediately if the condition that P_2 is to approach P_1 *along the circle* (1) is introduced.

Since the fixed point P_1 is on the circle (1), therefore

$$x_1^2+y_1^2=r^2; \quad . \quad . \quad . \quad (4)$$

and since P_2, while approaching P_1, always remains on circle (1), therefore

$$x_2^2+y_2^2=r^2; \quad . \quad . \quad . \quad (5)$$

hence, subtracting equation (4) from equation (5),

$$x_2^2-x_1^2+y_2^2-y_1^2=0,$$

that is, $(y_2-y_1)(y_2+y_1)=-(x_2-x_1)(x_2+x_1)$;

whence, $\dfrac{y_2-y_1}{x_2-x_1}=-\dfrac{x_2+x_1}{y_2+y_1}.$

Substituting this result in equation (2) gives

$$y-y_1=-\frac{x_2+x_1}{y_2+y_1}(x-x_1),^* \quad . \quad . \quad . \quad (6)$$

which is the equation of the secant line LM of the given circle (1).

---

* The difference between equations (2) and (6) consists in this: no matter where the points (x_1, y_1) and (x_2, y_2) may be, equation (2) represents the straight line passing through them; but equation (6) is the equation of the line through these points only when they are on the circle $x^2+y^2=r^2$. In other words, equation (2) is the equation of the line passing through any two points whatever, while equation (6) is the equation of the line passing through any two points on the circumference of the circle.

Now let P_2 move along the circle until it coincides with P_1, *i.e.*, until $x_2=x_1$, and $y_2=y_1$, then equation (6) becomes

$$y-y_1=-\frac{x_1+x_1}{y_1+y_1}(x-x_1),$$

i.e.,
$$y-y_1=-\frac{x_1}{y_1}(x-x_1),$$

which, by clearing of fractions and transposing, may be written in the form

$$x_1x+y_1y=x_1^2+y_1^2,$$

i.e.,
$$\boldsymbol{x_1x+y_1y=r^2}, \quad . \quad . \quad . \quad [35]$$

which is the required equation of the tangent to the circle $x^2+y^2=r^2$, x_1 and y_1 being the coördinates of the point of tangency.

(β) *Center of circle not at origin.* If the equation of the given circle be

$$x^2+y^2+2Gx+2Fy+C=0, \quad . \quad . \quad . \quad (7)$$

then, P_1 and P_2 being on this circle,

$$x_1^2+y_1^2+2Gx_1+2Fy_1+C=0, \quad . \quad . \quad . \quad (8)$$

and
$$x_2^2+y_2^2+2Gx_2+2Fy_2+C=0. \quad . \quad . \quad . \quad (9)$$

Subtracting equation (8) from equation (9),

$$x_2^2-x_1^2+2G(x_2-x_1)+y_2^2-y_1^2+2F(y_2-y_1)=0,$$

which may be written in the form

$$(y_2-y_1)(y_2+y_1+2F)=-(x_2-x_1)(x_2+x_1+2G);$$

whence,
$$\frac{y_2-y_1}{x_2-x_1}=-\frac{x_2+x_1+2G}{y_2+y_1+2F}.$$

Substituting this result in equation (2) gives

$$y-y_1=-\frac{x_2+x_1+2G}{y_2+y_1+2F}(x-x_1), \quad . \quad . \quad . \quad (10)$$

as the equation of the secant through the two points (x_1, y_1) and (x_2, y_2) on the circle (7). If, now, the point (x_2, y_2) moves along the curve until it comes into coincidence with (x_1, y_1), this secant line becomes a tangent, and its equation is

$$y - y_1 = -\frac{x_1 + G}{y_1 + F}(x - x_1). \quad . \quad . \quad . \quad (11)$$

Clearing equation (11) of fractions, and transposing, it may be written thus:

$$x_1x + y_1y + Gx + Fy = x_1^2 + y_1^2 + Gx_1 + Fy_1; \quad . \quad . \quad . \quad (12)$$

but, by equation (8), the second member of equation (12) equals

$$-Gx_1 - Fy_1 - C.$$

Putting this value for the second member in equation (12), ~~and~~ transposing, that equation becomes

$$\boldsymbol{x_1x + y_1y + G(x + x_1) + F(y + y_1) + C = 0}, \quad . \quad . \quad . \quad [36]$$

which is the required equation of the tangent to the circle (7), x_1 and y_1 being the coördinates of the point of contact.[*]

NOTE. Equation [36] may be easily remembered if it be observed that it differs from the equation of the circle [equation (7)] only in having x_1x, y_1y, $x + x_1$, and $y + y_1$ in place of x^2, y^2, $2x$, and $2y$, respectively. It will be found later that any equation of the second degree (from which the xy-term is absent) bears this same relation to the equation of a tangent to its locus, x_1 and y_1 being the coördinates of the point of contact. Compare, also, equation [35] with equation (1).

It must also be carefully kept in mind that equations [35] and [36] represent tangents *only if* (x_1, y_1) *is a point on the circle.* It will be seen later that these equations represent other lines if (x_1, y_1) is *not* on the circle.

85. Equation of a normal to a given circle. By definition (Art. 81) the normal at a given point, $P_1 \equiv (x_1, y_1)$, on any curve is the line through P_1, and perpendicular to the tangent at P_1. Hence, to get the equation of the normal at any given point, it is only necessary to write the equation of the tangent at this point (Art. 84), and then the equation of a line perpendicular to this tangent (Arts. 53, 62) and passing through the given point. Thus the equation of the normal to the circle

---

[*] Equations (11) and (12) are, of course, but different forms of the equation of the same tangent as that represented by equation [36].

$$x^2 + y^2 + 2Gx + 2Fy + C = 0, \quad . \quad . \quad . \quad (1)$$

at the point $P_1 \equiv (x_1, y_1)$, is

$$y - y_1 = \frac{y_1 + F}{x_1 + G}(x - x_1), \quad . \quad . \quad . \quad (2)$$

The coördinates $-G$ and $-F$ of the center of the given circle (1) satisfy equation (2); *hence, every normal to a circle passes through the center of the circle.*

If the center of the circle be at the origin, then $G = 0$, $F = 0$, and $C = -r^2$, and the equation (2) of the normal becomes

$$y - y_1 = \frac{y_1}{x_1}(x - x_1), \quad . \quad . \quad . \quad (3)$$

which reduces to $x_1 y - x y_1 = 0$,—an equation which could have been derived for the circle $x^2 + y^2 = r^2$ in precisely the same way that equation (2) was derived from equation (1).

EXERCISES

1. Derive, by the secant method, the equation of the tangent to the circle $x^2 + y^2 = 2rx$, the point of contact being $P_1 \equiv (x_1, y_1)$.

2. Write the equation of the tangent to the circle:

(α) $x^2 + y^2 = 25$, the point of contact being (3, 4);
(β) $x^2 + y^2 - 3x + 10y = 15$, the point of contact being (4, -11);
(γ) $(x - 2)^2 + (y - 3)^2 = 10$, the point of contact being (5, 4);
(δ) $3x^2 + 3y^2 - 2y - 4x = 0$, the point of contact being (0, 0).

3. Find the equation of the normal to each of the circles of Ex. 2, through the given point.

4. A tangent is perpendicular to the radius drawn to its point of contact. By means of this fact, derive the equation of the tangent to the circle $(x-a)^2+(y-b)^2=r^2$ at the point (x_1, y_1) (cf. equation [36]).

5. From the fact that a normal to a circle passes through its center, find the equation of the normal to the circle $x^2+y^2-6x+8y+21=0$ at the point $(1, {}^-4)$.

6. Find the equations of the two tangents, drawn through the external point (11, 3) to the circle $x^2+y^2=40$.

SUGGESTION. Use the equation of the tangent in terms of its slope.

7. What is the equation of the circle whose center is at the point (5, 3), and which touches the line $3x+2y-10=0$?

8. Under what condition will the line $\frac{x}{a}+\frac{y}{b}=1$ touch the circle $x^2+y^2=r^2$?

9. Find the equation of a circle inscribed in the triangle whose sides are the lines $x=0$, $y=0$, and $\frac{x}{a}+\frac{y}{b}=1$.

10. Solve Ex. 6 by assuming x_1 and y_1 as the coördinates of the point of contact, and then finding their numerical values from the two equations which they satisfy.

86. Lengths of tangents and normals. Subtangents and subnormals. The tangent and normal lines of any curve extend indefinitely in both directions; it is, however, convenient to consider as the **length of the tangent** the length TP_1, measured from the point of intersection (T) of the tangent with the x-axis to the point of tangency (P_1), and similarly to consider as the **length of the normal** the length P_1N, measured from P_1 to the point of intersection (N) of the normal with the x-axis.

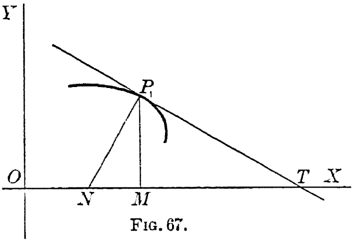


FIG. 67.

The **subtangent** is the length TM, where M is the foot of the ordinate of the point of tangency P_1; and the **subnormal** is the corresponding length MN. As thus taken, the subtangent and the subnormal are of the same sign; ordinarily, however, one is concerned merely with their *absolute* values, irrespective of the algebraic sign. The subtangent is the projection of the tangent length on the x-axis, and the subnormal is the like projection of the normal length.

87. Tangent and normal lengths, subtangent and subnormal, for the circle. The definitions given in the preceding article furnish a direct method for finding the tangent and normal lengths, as well as the subtangent and subnormal, for a circle. *E.g.*, to find these values for the circle $x^2+y^2=25$, and corresponding to the point of contact (3, 4), proceed thus:

The equation of the tangent P_1T is (Art. 84)

$$3x+4y=25;$$

hence the x-intercept of this tangent, *i.e.*, OT, $=\frac{25}{3}$; therefore the subtangent TM, which equals $OM-OT$, is $3-\frac{25}{3}$, *i.e.*, $-5\frac{1}{3}$. The tangent length

$$TP_1=\sqrt{\overline{MT}^2+\overline{MP_1}^2}=\sqrt{(\tfrac{16}{3})^2+4^2}=6\tfrac{2}{3}.$$

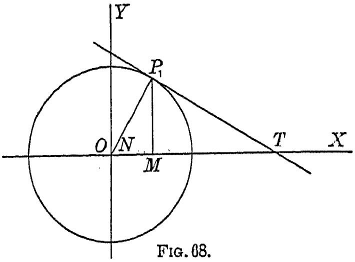


Fig. 68.

To find the normal length, and the subnormal, first write the equation of the normal at the point (3, 4); it is (Art. 85) $4x-3y=0$. Hence its x-intercept is zero, and the subnormal, MO in this case, is -3; the normal length P_1O is 5.

Similarly, corresponding to the point (x_1, y_1) on the circle $x^2+y^2=r^2$, the subtangent $=-\frac{y_1^2}{x_1}$, the tangent length $=\frac{ry_1}{x_1}$, the subnormal $=-x_1$, and the normal length $=r$. The derivation of these values is left as an exercise for the student, as is also the derivation of the corresponding expressions for the circle $x^2+y^2+2\,Gx+2\,Fy+C=0$, the point of contact being (x_1, y_1).

EXERCISES

Find the lengths of the tangent, subtangent, normal, and subnormal,

1. for the point $(4, {}^-11)$ on the circle $x^2+y^2-3\,x+10\,y=15$;

2. for the point $(1, 3)$ on the circle $x^2+y^2-10\,x=0$;

3. for the point whose abscissa is $\sqrt{7}$ on the circle $x^2+y^2=25$.

4. The subtangent for a certain point on a circle, whose center is at the origin, is $5\frac{1}{2}$, and its subnormal is 3. Find the equation of the circle, and the point of tangency.

88. To find the length of a tangent from a given external point to a given circle. Let $P_1\equiv(x_1, y_1)$ be the given external point, and let

$$x^2+y^2+2\,Gx+2\,Fy+C=0$$

be the given circle. The center of this circle (Art. 79) is $({}^-G, {}^-F)$, and its radius is $\sqrt{G^2+F^2-C}$. Join P_1 to the center K, draw the tangent P_1Q, and also the radius KQ. Then $\overline{P_1Q}^2=\overline{KP_1}^2-\overline{KQ}^2$;

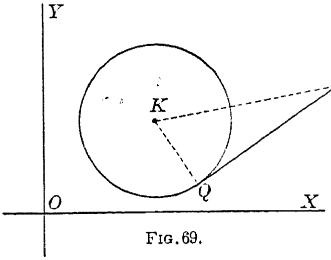


FIG. 69.

but

$$\overline{KP_1}^2=(x_1+G)^2+(y_1+F)^2. \quad \text{(Art. 26)}$$

and $$\overline{KQ}^2=G^2+F^2-C; \quad \text{(Art. 79)}$$

$$\therefore \quad P_1Q^2 = (x_1 + G)^2 + (y_1 + F)^2 - (G^2 + F^2 - C)$$
$$= x_1^2 + y_1^2 + 2\,Gx_1 + 2\,Fy_1 + C,$$

i.e., the square of the length of the tangent from a given external point to the circle $x^2 + y^2 + 2\,Gx + 2\,Fy + C = 0$* *is obtained by writing the first member only of this equation, and substituting in it the coördinates of the given point.*†

89. From any point outside of a circle two tangents to the circle can be drawn. (*a*) Let the equation of the circle be

$$x^2 + y^2 = r^2, \quad . \quad . \quad . \quad (1)$$

then (Art. 83) the line

$$y = mx + r\sqrt{1 + m^2} \quad . \quad . \quad . \quad (2)$$

is, for all values of m, tangent to this circle. Let $P_1 \equiv (x_1, y_1)$ be any given point outside the circle (1); then the tangent (2) will pass through P_1 if, and only if, m be given a value such that the equation

$$y_1 = mx_1 + r\sqrt{1 + m^2} \quad . \quad . \quad . \quad (3)$$

shall be satisfied.

Transposing, squaring, and rearranging equation (3), it is clear that it will be satisfied if, and only if, m is given a value such that the equation

$$(r^2 - x_1^2)m^2 + 2\,x_1y_1m + r^2 - y_1^2 = 0$$

is satisfied; *i.e.*, equation (3) is satisfied if, and only if,

$$m = \frac{-x_1y_1 \pm r\sqrt{x_1^2 + y_1^2 - r^2}}{r^2 - x_1^2}. \quad . \quad . \quad . \quad (4)$$

Equation (4) gives *two*, and only two, real values for m when (x_1, y_1) is *outside* of the circle, for then $x_1^2 + y_1^2 - r^2$ is positive (Art. 78, foot-note); these values of m, being substituted in turn in equation (2), give the two tangents through P_1 to the circle (1).

---

* If the circle is given by the equation $Ax^2 + Ay^2 + 2\,Gx + 2\,Fy + C = 0$, it must first be divided by A before applying this theorem.

† The expression $x_1^2 + y_1^2 + 2\,Gx_1 + 2\,Fy_1 + C$ is called the *power of the point* $P_1 \equiv (x_1, y_1)$ with regard to the circle $x^2 + y^2 + 2\,Gx + 2\,Fy + C = 0$.

If P_1 is *on* the circle (1), then $x_1^2 + y_1^2 - r^2 = 0$; hence the two values of m from equation (4) coincide, and the two tangents also coincide, *i.e.*, there is in this case but *one* tangent. If P_1 is within the circle, then the two values of m from equation (4) are both imaginary and no tangent through P_1 can be drawn to the circle (1).*

If either value of m from equation (4) is substituted in equation (2), and then equations (2) and (1) are considered as simultaneous and solved for x and y, the coördinates of the corresponding point of contact are obtained.

NOTE. The properties of the *equations* of the line and circle have thus established a geometric property of the circle [cf. Art. 31, (III)].

(β) If the equation of the given circle had been

$$x^2 + y^2 + 2\,Gx + 2\,Fy + C = 0, \quad . \quad . \quad . \quad (5)$$

it could, by Art. 71, have been transformed to new axes through its center $({}^{-}G, {}^{-}F)$ and parallel respectively to the given axes; its equation would thus have become

$$x'^2 + y'^2 = r^2, \quad . \quad . \quad . \quad (6)$$

where x' and y' refer to the new axes.

This transformation, however, leaves the circle and all its intrinsic properties unchanged; but (a) applies to circle (6), hence it is proved that circle (5), which is circle (6) merely referred to other axes, has the same properties.

---

* These conclusions may also be stated thus: if P_1 is *outside* of the circle, equation (4) gives two real and distinct values for m; corresponding to these there are two real and distinct tangents; if P_1 is *on* the circle, the two values of m are real but coincident, and there are two real but coincident tangents; if P_1 is *inside* of the circle, the two values of m are imaginary, and the two corresponding tangents are therefore also imaginary.

90. Chord of contact. If two tangents are drawn from any external point to a circle, the line joining the two corresponding points of tangency is called the **chord of contact** for the point from which the tangents are drawn.

The equation of this chord of contact may be found by first finding the points of tangency and then writing the equation of the straight line through those two points. It may, however, be found more briefly, and much more elegantly, as follows:

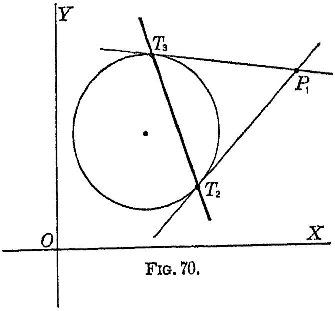


FIG. 70.

Let $P_1 \equiv (x_1, y_1)$ be the given external point from which the two tangents are drawn; and let $T_2 \equiv (x_2, y_2)$ and $T_3 \equiv (x_3, y_3)$ be the points of tangency on the circle

$$x^2 + y^2 + 2\,Gx + 2\,Fy + C = 0; \quad . \quad . \quad . \quad (1)$$

it is required to find the equation of the line passing through T_2 and T_3. The equation of the tangent at T_2 is (Art. 84)

$$x_2x + y_2y + G(x + x_2) + F(y + y_2) + C = 0, \ . \ . \ . \ (2)$$

and the equation of the tangent at T_3 is

$$x_3x + y_3y + G(x + x_3) + F(y + y_3) + C = 0. \ . \ . \ . \ (3)$$

But each of these tangents passes through the point P_1; hence its coördinates, x_1 and y_1, satisfy equations (2) and (3), therefore

$$x_1x_2 + y_1y_2 + G(x_1 + x_2) + F(y_1 + y_2) + C = 0, \ . \ . \ . \ (4)$$

and $$x_1x_3 + y_1y_3 + G(x_1 + x_3) + F(y_1 + y_3) + C = 0. \ . \ . \ . \ (5)$$

Equations (4) and (5), however, assert respectively that (x_2, y_2) and (x_3, y_3) are points on the locus of the equation

$$x_1x + y_1y + G(x_1 + x) + F(y_1 + y) + C = 0. \ . \ . \ . \ (6)$$

But equation (6) is of the first degree in the two variables x and y, hence (Art. 57) its locus is a straight line, and, since it passes through both $T_1 \equiv (x_2, y_2)$ and $T_3 \equiv (x_3, y_3)$, it is the equation of the chord of contact;

i.e., $$x_1x + y_1y + G(x + x_1) + F(y + y_1) + C = 0 \quad . \; . \; . \quad [37]$$

is the equation of the chord of contact corresponding to the external point $P_1 \equiv (x_1, y_1)$.

It is to be noticed that if P_1 is *on* the circle, then the two tangents drawn through it coincide with each other and with the chord of contact; the equation of the chord of contact [37] then becomes the equation of the tangent at P_1, as it should (cf. equation [36]).

If, then, (x_1, y_1) is a point on the circle (1), equation [37] is the equation of the tangent to the circle at that point; if, on the other hand, (x_1, y_1) is outside of this circle, then equation [37] is not the equation of a tangent, but of the chord of contact corresponding to that external point.

EXERCISES

1. Find the length of the tangent from the point (8, 10) to the circles:

(α) $x^2 + y^2 - 3x = 0$; (β) $2x^2 + 2y^2 = 5y + 6$.

2. (α) Write the equation of the chord of contact corresponding to the point (5, 6) for the circle $x^2 + y^2 - 6x - 4y = 8$.

(β) Find the coördinates of the points in which this chord cuts the circle.

(γ) Write the equations of the tangents to the circle at these points of intersection; show that these lines pass through the given point (5, 6).

3. By the method of exercise 2, find the equations of the tangents drawn to the circle $(3x - 2)^2 + (3y + 5)^2 = 4$, from the origin; from the point (1, 2).

4. Find the locus of a point from which the tangents drawn to the two circles

$$2x^2 + 2y^2 - 10x + 14y + 35 = 0 \quad \text{and} \quad x^2 + y^2 = 9$$

are of equal length. Show that this locus is a straight line perpendicular to the line joining the centers of the given circles.

5. For what point is the line $3x+4y=7$ the chord of contact with regard to the circle $x^2+y^2=14$?

6. Find the chord of contact for the circle $x^2+y^2=25$, corresponding to the point (3, 7); to the point (3, 2).

7. By means of the equation $y-y_1=m(x-x_1)$ prove that two tangents can be drawn through the external point (x_1, y_1) to the circle whose equation is $x^2+y^2=r^2$.

8. Solve (β) and (γ), of exercise 2, by means of the equation

$$y-6=m(x-5).$$

91. Poles and Polars. If through any given point $P_1 \equiv (x_1, y_1)$, outside, inside, or on the circle, a secant is drawn, meeting the circle in two points, as Q and R, and if tangents are drawn at Q and R, they will intersect in some point as

$$P' \equiv (x', y').$$

The locus of P', as the secant revolves about P_1, is called the **polar** of P_1 with regard to the circle; and P_1 is the **pole** of that locus. It will be proved in the next article that the locus of P' is a straight line whose equation is of the same form as that of the tangent (Art. 84), and as that of the chord of contact (Art. 90) already found.

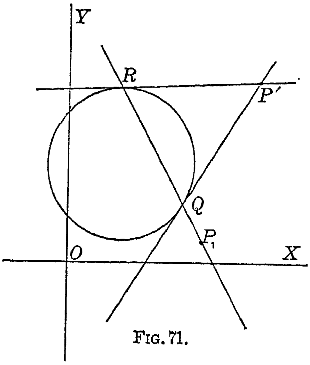


FIG. 71.

92. Equation of the polar. Let $P_1 \equiv (x_1, y_1)$ be the given point, the equation of whose polar, with regard to the circle

$$x^2+y^2+2Gx+2Fy+C=0, \quad . \quad . \quad . \quad (1)$$

is sought. Also let P_1QR be any position of the secant through P_1, and let the tangents at Q and R intersect in $P' \equiv (x', y')$; then the equation of P_1QR (Art. 90) is

$$x'x+y'y+G(x+x')+F(y+y')+C=0. \quad . \quad . \quad . \quad (2)$$

Since P_1 is on this line, therefore

$$x_1x' + y_1y' + G(x_1 + x') + F(y_1 + y') + C = 0. \quad . \quad . \quad . \quad (3)$$

Equation (3) asserts that the coördinates, x' and y', of P' satisfy the equation

$$x_1x + y_1y + G(x + x_1) + F(y + y_1) + C = 0; \quad . \quad . \quad . \quad [38]$$

i.e., this variable point P' always lies on the locus of equation [38]; in other words, [38] is the equation of the polar of P_1 with regard to the circle (1).

Moreover, since equation [38] is of the first degree in the variables x and y, therefore (Art. 57) its locus is a straight line; that is, *the polar of any given point, with regard to any given circle, is a straight line.*

That equations [36] and [37] have the same form as equation [38] is due to the fact that the tangent and the chord of contact are only special cases of the polar.

93. Fundamental theorem. An important theorem concerning poles and polars is: *If the polar of the point P_1, with regard to a given circle, passes through the point P_2, then the polar of P_2 passes through P_1.* Let the equation of the given circle be

$$x^2 + y^2 + 2\,Gx + 2\,Fy + C = 0, \quad . \quad . \quad . \quad (1)$$

and let the two given points

be $\qquad P_1 \equiv (x_1, y_1)$,

and $\qquad P_2 \equiv (x_2, y_2)$;

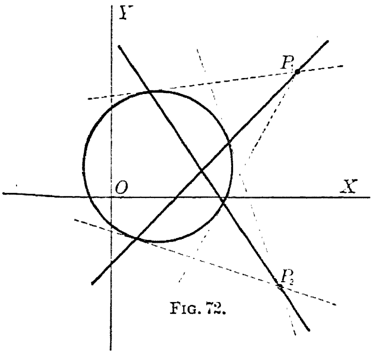


FIG. 72.

then (Art. 92) the equation of the polar of P_1 is

$$x_1x + y_1y + G(x + x_1) + F(y + y_1) + C = 0. \quad . \quad . \quad . \quad (2)$$

If this line passes through P_2, then

$$x_1x_2 + y_1y_2 + G(x_2 + x_1) + F(y_2 + y_1) + C = 0. \quad . \quad . \quad . \quad (3)$$

But the equation of the polar of P_2 (Art. 92) is

$$x_2x + y_2y + G(x + x_2) + F(y + y_2) + C = 0, \quad . \quad . \quad . \quad (4)$$

and equation (3) proves that the locus of equation (4) passes through P_1, which establishes the theorem.

EXERCISES

1. Find the polar of the point (6, 8) with reference to the circle $x^2 + y^2 = 14$.

2. Find the polar of the point (1, 2) with regard to the circle $x^2 + y^2 + 4x - 6y = 10$.

3. Find the pole of the line $4x + 6y = 7$, and of the line $ax + by - 1 = 0$, with regard to the circle $x^2 + y^2 = 35$.

4. Find the equations of the two tangents to the circle $x^2 + y^2 = 65$ from the point (4, 7); from the point (11, 3).

5. Show that if the polar of (h, k) with respect to the circle $x^2 + y^2 = c^2$ touch the circle $4(x^2 + y^2) = c^2$, then the pole (h, k) will lie on the circle $x^2 + y^2 = 4c^2$.

6. Show that the pole of the line joining (5, 7) and (−11, 1) is the point of intersection of the polars of those two points with reference to the circle $x^2 + y^2 = 100$.

7. Find the pole of the line $2x - 3y = 0$ with respect to the circle $x^2 + y^2 = 9$.

8. Show what specialization of a polar converts it into a chord of contact, and what further specialization converts it into a tangent.

94. Geometrical construction for the polar of a given point, and for the pole of a given line, with regard to a given circle. Since the relation between a polar and its pole (see def. Art. 91) is independent of the coördinate axes, therefore the given circle may, without loss of generality, be assumed to have its center at the origin.

If $P_1 \equiv (x_1, y_1)$ is any given point, and

$$x^2 + y^2 = r^2 \quad . \quad . \quad . \quad (1)$$

is a given circle, whose center is at the point O, then the equation of OP_1 (Art. 51) is

$$y_1x - x_1y = 0. \quad . \quad . \quad . \quad . \qquad (2)$$

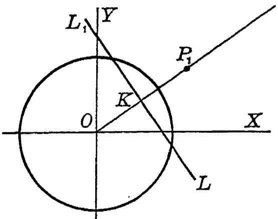


FIG. 73.a

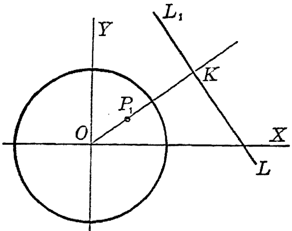


FIG. 73.b

Let LL_1 be the polar of P_1, with regard to the given circle, and let it meet OP_1 in K. The equation of LL_1 (Art. 92) is

$$x_1x + y_1y = r^2. \quad . \quad . \quad . \quad . \qquad (3)$$

Equations (2) and (3) show (Art. 62) that LL_1 and OP_1 are perpendicular to each other; *i.e.*, *the line joining the given point* P_1 *to the center of the circle is perpendicular to the polar of* P_1 *with regard to the circle.*

The distance (OK) from the origin to the line LL_1 (Art. 64) is

$$\frac{r^2}{\sqrt{x_1^2 + y_1^2}}, \quad . \quad . \quad . \quad . \qquad (4)$$

and the length of OP_1 (Art. 26) is

$$\sqrt{x_1^2 + y_1^2}; \quad . \quad . \quad . \quad . \qquad (5)$$

therefore $\quad OK \cdot OP_1 = \dfrac{r^2}{\sqrt{x_1^2 + y_1^2}} \cdot \sqrt{x_1^2 + y_1^2} = r^2.$

Hence, to construct, with regard to a given circle, the polar of any given point P_1, join that point to the center of the circle, then on OP_1 (produced if necessary) find a point K such that the rectangle $OP_1 \cdot OK$ is equal to the square on the radius of the circle, and through K draw a line perpendicular to OP_1; this line is the required polar.

Similarly the pole may be constructed, if the polar and the circle are given.

95. Circles through the intersections of two given circles. Given two circles whose equations are

$$x^2 + y^2 + 2G_1x + 2F_1y + C_1 = 0, \quad . \quad . \quad . \quad (1)$$

and

$$x^2 + y^2 + 2G_2x + 2F_2y + C_2 = 0. \quad . \quad . \quad . \quad (2)$$

These circles intersect, in general, in two finite points $P_1 \equiv (x_1, y_1)$ and $P_2 \equiv (x_2, y_2)$, and (Art. 41) the equation

$$x^2 + y^2 + 2G_1x + 2F_1y + C_1 + k(x^2 + y^2 + 2G_2x + 2F_2y + C_2) = 0, \quad . \quad . \quad . \quad (3)$$

where k is any constant, represents a curve which passes through these same points P_1 and P_2.

The locus of equation (3) is, moreover, a circle (Art. 79); hence, a series of different values being assigned to the parameter k, equation (3) represents what is called a "family" of circles; each one of these circles passing through the two points P_1 and P_2 in which the given circles (1) and (2) intersect each other.

96. Common chord of two circles. If in equation (3), Art. 95, the parameter k be given the particular value -1, the equation reduces to

$$2(G_1 - G_2)x + 2(F_1 - F_2)y + C_1 - C_2 = 0, \quad . \quad . \quad . \quad (4)$$

which is of the first degree, and therefore represents a straight line; but this locus belongs to the family represented by equation (3) of Art. 95, hence it passes through the two points P_1 and P_2 in which the circles (1) and (2) intersect. This line (4) is, therefore, the **common chord** * of these circles.

To obtain the equation of the common chord of two given circles it is, then, only necessary to eliminate the terms in x^2 and y^2 between their equations. *E.g.*, to find the common chord of the circles

$$2x^2 + 2y^2 + 3x + 5y - 9 = 0, \quad . \quad . \quad . \quad (\alpha)$$

and

$$6x^2 + 6y^2 + 11x + 13y - 23 = 0, \quad . \quad . \quad . \quad (\beta)$$

multiply equation (α) by 3 and subtract the result from equation (β); this gives

$$x - y + 2 = 0, \quad . \quad . \quad . \quad (\gamma)$$

as the equation of the common chord of the given circles.

This result may be verified by finding the points of intersection (Art. 39) of the circles (α) and (β), and then writing the equation of the straight line through those two points.

Since the common chord of two circles intersects each of these circles in the points in which they intersect each other, therefore the points of intersection of two circles may be found by finding the points in which their common chord intersects either of them. *E.g.*, to find the points in which the circles (α) and (β) intersect each other, it is only necessary to find the points in which (γ) cuts either (α) or (β).

97. Radical axis; radical center. The line whose equation is obtained by eliminating the x^2 and y^2 terms between the equations of two given circles, as in Art. 96, whether the circles intersect in real points or not, is called the **radical axis** of the two circles. If the two given circles intersect each other in real points, then this line is also called their common chord; that is, the common chord of two circles is a special case of the radical axis of two circles.

---

* Equation (3) of Art. 95, which for every value of k represents a circle passing through the two points in which the given circles (1) and (2) intersect, may be written in the form

$$x^2 + y^2 + 2\frac{G_1 + kG_2}{1 + k}x + 2\frac{F_1 + kF_2}{1 + k}y + \frac{C_1 + kC_2}{1 + k} = 0.$$

The coördinates of the center of this circle are (Art. 79)

$$-\frac{G_1 + kG_2}{1 + k} \quad \text{and} \quad -\frac{F_1 + kF_2}{1 + k}.$$

If then k be made to approach -1, both of these coördinates approach infinity, but the circle always passes through the two fixed points in which the given circles intersect; hence the common chord of two given circles may be regarded as an infinitely large circle whose center is at infinity.

Three circles, taken two and two, have three radical axes. It is easily shown that these three radical axes pass through a common point; this point is called the **radical center** of the three circles.

EXERCISES

1. Find the equation of the common chord of the circles

$$x^2 + y^2 - 3x - 5y - 8 = 0, \quad x^2 + y^2 + 8x = 0.$$

2. Find the point of intersection of the circles in exercise 1, and the length of their common chord.

3. Find the radical axis, and also the length of the common chord, for the circles $x^2 + y^2 + ax + by + c = 0$, $x^2 + y^2 + bx + ay + c = 0$.

4. Find the radical center of the three circles

$$x^2 + y^2 + 4x + 7 = 0,$$
$$2(x^2 + y^2) + 3x + 5y + 9 = 0,$$
$$x^2 + y^2 + y = 0.$$

5. Show that tangents from the radical center, in exercise 4, to the three circles, respectively, are equal in length.

6. Prove analytically that the tangents to two circles from any point on their radical axis are equal.

7. Find the polar of the radical center of the circles in exercise 4, with respect to each circle.

8. Prove analytically that the three radical axes of three circles, the circles being taken in pairs, meet in a common point.

98. The equation of a circle: polar coördinates. Let OR be the initial line, O the pole, $C \equiv (\rho_1, \theta_1)$ the center of the circle, r its radius, and $P \equiv (\rho, \theta)$ any point on the circle. Draw OC, OP, and CP; then, by trigonometry,

$$r^2 = \rho^2 + \rho_1^2 - 2\rho\rho_1 \cos(\theta - \theta_1), \text{ i.e.,}$$

$$\rho^2 - 2\rho_1\rho \cos(\theta - \theta_1) + \rho_1^2 - r^2 = 0, \quad . \quad . \quad . \quad [39]$$

which is the equation of the given circle.

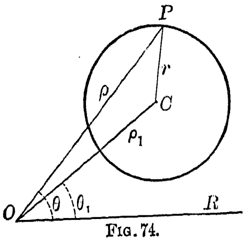


FIG. 74.

Depending upon the relative positions of the polar axis, the pole, and the center of the circle, equation [39] has several special forms :

(α) If the center is on the polar axis, then $\theta_1 = 0$, and equation [39] becomes

$$\rho^2 - 2\rho_1\rho\cos\theta + \rho_1^2 - r^2 = 0;$$

(β) If the pole is on the circle, then $\rho_1 = r$, and equation [39] becomes

$$\rho - 2r\cos(\theta - \theta_1) = 0;$$

(γ) If the pole is on the circle and the polar axis a diameter, then $\rho_1 = r$ and $\theta_1 = 0$, and equation [39] becomes

$$\rho - 2r\cos\theta = 0;$$

(δ) If the center is at the pole, then $\rho_1 = 0$ and equation [39] becomes

$$\rho = r.$$

99. Equation of a circle referred to oblique axes. Let the axes OX and OY be inclined at an angle ω; let $C \equiv (h, k)$ be the center of the circle, r its radius, and $P \equiv (x, y)$ any point on the circle. Draw the ordinates M_1C and MP, connect C and P, and draw CHL parallel to the x-axis; then

$$\overline{CP}^2 = \overline{CH}^2 + \overline{HP}^2 + 2\,CH\cdot HP\cos\omega;$$

hence $r^2 = (x-h)^2 + (y-k)^2 + 2(x-h)(y-k)\cos\omega$,

i.e., $(x-h)^2 + (y-k)^2 + 2(x-h)(y-k)\cos\omega - r^2 = 0;$ . . . [40]

which is the equation of the given circle.

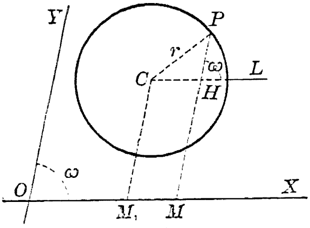


FIG. 75.

It is to be observed that this equation [40] is not of the form

$$x^2 + y^2 + 2Gx + 2Fy + C = 0,$$

which was discussed in Art. 79; it differs from that equation in that it contains an xy-term. If, however, the axes are rectangular, as in Art. 79, then $\cos\omega = 0$, and equation [40] reduces to the standard form of Art. 79, viz.:

$$x^2 + y^2 + 2Gx + 2Fy + C = 0,$$

which is a special case of equation [40].

100. The angle formed by two intersecting curves. By the angle between two intersecting curves is meant the angle formed by the two tangents, one to each curve, drawn through the point of intersection.

Hence to find the angle at which two curves intersect, it is only necessary to find the point of intersection, then to find the equations of the tangents at this point, one to each curve, and finally to find the angle formed by these tangents.

EXERCISES

1. Find the polar equation of the circle whose center is at the point $\left(7, \frac{\pi}{4}\right)$ and whose radius is 10; determine also the points of its intersection with the initial line.

2. Find the polar equation of a circle whose center is at the point $\left(15, \frac{\pi}{2}\right)$ and whose radius is 10. Find also the equations of the tangents to the circle from the pole.

3. A circle of radius 3 is tangent to the two radii vectores which make the angles 60° and 120° with the initial line: find its polar equation, and the distance of the center from the origin.

4. Find the equation of a circle of radius 5, with center at the point (2, 3), if ω is 60°.

5. Find the equation of a circle of radius 2, with center at the origin, if ω is 120°.

6. Determine the equation of the circle circumscribing an equilateral triangle, — the coördinate axes being two sides of the triangle.

7. A circle is inscribed in a square. What is its equation, if a side and adjacent diagonal of the square are chosen as the y- and x-axis, respectively? What are the coördinates of the points of tangency?

8. Find the angle at which the circle $x^2 + y^2 = 9$ intersects the circle $(x - 4)^2 + y^2 - 2y = 15$. At what angle does the second of these circles meet the line $x + 2y = 4$?

EXAMPLES ON CHAPTER VII

1. Find the equation of the circle circumscribing the triangle whose vertices are at the points $(7, 2)$, $({}^-1, {}^-4)$, and $(3, 3)$. What is its center? its radius?

2. Determine the center of the circle

$$(x + a)^2 + (y + b)^2 = a^2 + b^2.$$

What family of circles is represented by this equation, if a and b vary under the one restriction that $a^2 + b^2$ is to remain constant?

3. What must be the relations among the coefficients in order that the circles

$$x^2 + y^2 + 2G_1x + 2F_1y + C_1 = 0,$$

and

$$x^2 + y^2 + 2G_2x + 2F_2y + C_2 = 0,$$

shall be concentric? that they shall have equal areas?

4. Under what limitations upon the coefficients is the circle

$$Ax^2 + Ay^2 + Dx + Ey + F = 0$$

tangent to each of the axes?

5. Find the equation of the circle which has its center on the x-axis, and which passes through the origin and also through the point $(2, 3)$.

6. Find the points of intersection of the two circles

$$x^2 + y^2 - 4x - 2y - 31 = 0 \text{ and } x^2 + y^2 - 4x + 2y + 1 = 0.$$

7. Circles are drawn having their centers at the vertices of the triangle $(7, 2)$, $({}^-1, {}^-4)$ and $(3, 3)$, respectively, and each passing through the center of a fourth circle which circumscribes this triangle; find their equations, their common chords, and their radical center.

8. Circles having the sides of the triangle $(7, 2)$, $({}^-1, {}^-4)$, $(3, 3)$ as diameters are drawn; find their equations, their radical axes, and their radical center.

9. Find the equation of the circle passing through the origin and the point (x_1, y_1), and having its center on the y-axis.

10. The point $(3, -5)$ bisects a chord of the circle $x^2 + y^2 = 277$; find the equation of that chord.

11. A circle touches the line $4x + 3y + 3 = 0$ at the point $(-3, 3)$ and passes through the point $(5, 9)$; find its equation.

12. A circle, whose center coincides with the origin, touches the line $7x - 11y + 2 = 0$; find its equation.

13. At the points in which the circle $x^2 + y^2 - ax - by = 0$ cuts the axes, tangents are drawn; find the equations of these tangents.

14. A circle, whose radius is 7, touches the line $5y = 7x - 1$ at the point $(8, 11)$; find the equation of this circle.

15. A circle is inscribed in the triangle $(7, 2)$, $(-1, -4)$, $(3, 3)$; find its equation; find also the equations of the polars of the three vertices with regard to this circle.

16. Through a fixed point (x_1, y_1) a secant line is drawn to the circle $x^2 + y^2 = r^2$; find the locus of the middle point of the chord which the circle cuts from this secant line, as the secant revolves about the given fixed point (x_1, y_1).

17. Prove analytically that an angle inscribed in a semicircle is a right angle.

18. Prove analytically that a radius drawn perpendicular to a chord of a circle bisects that chord.

19. Show that the distances of two points from the center of a circle are proportional to the distances of each from the polar of the other.

20. Two straight lines touch the circle $x^2 + y^2 - 5x - 3y + 6 = 0$, one at the point $(1, 1)$ and the other at the point $(2, 3)$; find the pole of the chord of contact of these tangents.

21. Find the condition among the coefficients that must be satisfied if the circles

$$x^2 + y^2 + 2G_1x + 2F_1y = 0 \quad \text{and} \quad x^2 + y^2 + 2G_2x + 2F_2y = 0$$

shall touch each other at the origin.

22. Determine G, F, and C so that the circle

$$x^2 + y^2 + 2Gx + 2Fy + C = 0$$

shall cut each of the circles

$$x^2 + y^2 - 4x - 2y + 4 = 0 \quad \text{and} \quad x^2 + y^2 + 4x + 2y = 1$$

at right angles (cf. Art. 100).

23. Given the two circles

$$x^2+y^2-4x-2y+4=0 \quad \text{and} \quad x^2+y^2+4x+2y-4=0;$$

find the equation of their common tangents.

24. Find the radical axis of the circles in example 23; show that it is perpendicular to the line joining the centers of the given circles, and find the ratio of the lengths of the segments into which the radical axis divides the line joining the centers. How is this ratio related to the radii of the circles? Is this relation true for any pair of circles whatever?

25. Given the three circles:

$$x^2+y^2-16x+60=0, \qquad 3x^2+3y^2-36x+81=0,$$

and

$$x^2+y^2-16x-12y+84=0;$$

find the point from which tangents drawn to these three circles are of equal length, also find that length. How is this point related in position to the radical center of the given circles? Prove that this relation is the same for any three circles.

26. Find the locus of a point which moves so that the length of the tangent, drawn from it to a fixed circle, is in a constant ratio to the distance of the moving point from a given fixed point.

27. Let P be a fixed point on a given circle, T a point moving along the circle, and Q the point of intersection of the tangent at T with the perpendicular upon it from P; find the locus of Q.

SUGGESTION. Use polar coördinates, P being the pole, and the diameter through P the initial line.

28. Find the length of the common chord of the two circles

$$(x-a)^2+(y-b)^2=r^2 \quad \text{and} \quad (x-b)^2+(y-a)^2=r^2.$$

From this find the condition that these circles shall touch each other.

29. If the axes are inclined at 60°, prove that the equation

$$x^2+xy+y^2-4x-5y-2=0$$

represents a circle; find its radius and center.

30. What is the obliquity of the axes if the equation

$$x^2+\sqrt{3}\,xy+y^2-4x-6y+5=0$$

represents a circle? What is its radius?

31. For what point on the circle $x^2+y^2=9$ are the subtangent and the subnormal of equal length? the tangent and normal? the tangent and subtangent?

32. An equilateral triangle is inscribed in the circle $x^2 + y^2 = 4$ with its base parallel to the x-axis; through its vertices tangents to the circle are drawn, thus forming a circumscribed triangle; find the equations, and the lengths, of the sides of each triangle.

33. The poles of the sides of each triangle in example 32 are the vertices of a triangle; find the equations of its sides, and draw the figure.

34. A chord of the circle $x^2 + y^2 - 22x - 4y + 25 = 0$ is of length $4\sqrt{5}$, and is parallel to the line $2x + y + 7 = 0$; find the equation of the chord, and of the normals at its extremities.

35. Find the equation of a circle through the intersection of the circles $x^2 + y^2 - 4 = 0$, $x^2 + y^2 - 2x - 4y + 5 = 0$, and tangent to the line $x + y - 3 = 0$.

36. The length of a tangent, from a moving point, to the circle $x^2 + y^2 = 6$ is always twice the length of the tangent from the same point to the circle $x^2 + y^2 + 3(x + y) = 0$. Find the equation of the locus of the moving point.

37. Find the locus of the vertex of a triangle having given the base $= 2a$, and the sum of the squares of its sides $= 2b^2$.

38. Find the locus of the middle points of chords drawn through a fixed point on the circle $x^2 + y^2 = a^2$.

39. Through the external point $P_1 \equiv (x_1, y_1)$, a line is drawn meeting the circle $x^2 + y^2 = a^2$ in Q and R; find the locus of middle point of P_1Q as this line revolves about P_1.

40. A point moves so that its distance from the point $(1, 3)$ is to its distance from the point $(^-4, 1)$ in the ratio $2:3$. Find the equation of its locus.

41. Do the circles

$$4x^2 + 4y^2 + 4x - 12y + 1 = 0 \quad \text{and} \quad 2x^2 + 2y^2 + y = 0$$

intersect? Show in two ways.

42. Find the equation of a circle of radius $\sqrt{85}$ which passes through the points $(2, 1)$ and $(^-3, 4)$.

43. What are the equations of the tangent and the normal to the circle $x^2 + y^2 = 13$, — these lines passing through the point $(2, ^-3)$? through the point $(0, 6)$?

44. Find the equations of the tangents through $(2, 3)$ to the circle

$$9(x^2 + y^2) + 6x - 12y + 4 = 0.$$

45. At what angle do the circles $x^2+y^2+6x-2y+5=0$ and $x^2+y^2+4x+2y-5=0$ intersect each other?

46. A diameter of the circle $4x^2+4y^2+8x-12y+1=0$ passes through the point $(1, -1)$. Find its equation, and the equation of the chords which it bisects.

47. Find the locus of a point such that tangents from it to two concentric circles are inversely proportional to the radii of the circles.

48. Find the locus of a point which moves so that its distances from two fixed points are in constant ratio k. Discuss the locus and draw the figure.

49. A point moves so that the square of its distance from the base of an isosceles triangle is equal to the product of its distances from the other two sides. Show that the locus is a circle.

50. Prove that the two circles

$$x^2+y^2+2G_1x+2F_1y+C_1=0 \text{ and } x^2+y^2+2G_2x+2F_2y+C_2=0$$

are concentric if $G_1=G_2$ and $F_1=F_2$; that they are tangent to each other if

$$\sqrt{(G_1-G_2)^2+(F_1-F_2)^2}=\sqrt{G_1^2+F_1^2-C_1}\pm\sqrt{G_2^2+F_2^2-C_2};$$

and find the condition among the constants that these circles intersect orthogonally, *i.e.*, at right angles to each other.

CHAPTER VIII

THE CONIC SECTIONS

101. In Art. 48, which should now be carefully re-read, a conic section was defined; its general equation was derived; its three species, viz., the parabola, ellipse, and hyperbola, were mentioned; and a brief discussion of the nature and forms of the curve was given. In the present chapter, each of these three species will be examined somewhat more closely than was done in Chapter IV, and some general theorems concerning its tangents, normals, diameters, chords of contact, and polars will be proved.

The general equation (Art. 48) of the conic section might here be assumed, and the special forms for the parabola, the ellipse, and the hyperbola be derived from it; but, partly as an exercise, and partly for the sake of freedom to choose the axes in the most advantageous ways, the equations will here be re-derived, as they are needed, from the definitions of the curves.

I. THE PARABOLA

Special Equation of Second Degree

$Ax^2 + 2Gx + 2Fy + C = 0$, or $By^2 + 2Gx + 2Fy + C = 0$

102. The parabola defined. A parabola is the locus of a point which moves so that its distance from a fixed point, called the **focus**, is equal to its distance from a fixed line called the **directrix.** It is the conic section with eccentricity $e = 1$ (cf. Art. 48).

The equation of a parabola, with any given focus and directrix, can be obtained directly from this definition.

EXAMPLE. To find the equation of the parabola whose directrix is the line $x - 2y - 1 = 0$, and whose focus is the point $(2, -3)$.

Let $P \equiv (x, y)$ be any point on the parabola (see Fig. 79);

then $\dfrac{x - 2y - 1}{+\sqrt{5}}$ is the distance of P from the directrix (Art. 64),

and $\sqrt{(x-2)^2 + (y+3)^2}$ is the distance of P from the focus (Art. 26);

hence $$\frac{x - 2y - 1}{+\sqrt{5}} = \sqrt{(x-2)^2 + (y+3)^2},\text{ by definition;}$$

that is, $$4x^2 + 4xy + y^2 - 18x + 26y + 64 = 0;$$

which is the required equation.

The equation obtained in this way is not, however, in the most suitable form from which to study the properties of the curve, but can be simplified by a proper choice of axes. In Art. 48 it was shown that the parabola is symmetrical with respect to the straight line through the focus and perpendicular to the directrix, and that it cuts this line in only one point. If this line of symmetry is taken as the x-axis, the equation will have no y-term of first degree [cf. Art. 48, eq. (3)]; while if the point of intersection of the curve with this axis be taken as origin, the equation will have no constant term, since the point $(0, 0)$ must satisfy the equation. With this choice of axes, the equation of the parabola will reduce to a simple form, which is usually called the *first standard equation* of the parabola.

103. First standard form of the equation of the parabola. Let $D'D$ be the directrix of the parabola, and F its focus;

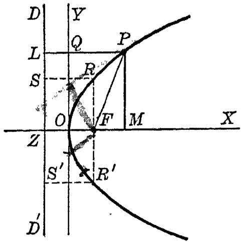


FIG. 76.

also let the line ZFX, perpendicular to the directrix, be the x-axis; denote the fixed distance ZF by $2p$, and let O, its middle point, be the origin of coördinates; then the line OY, perpendicular to OX, is the y-axis. Let $P \equiv (x, y)$ be any point on the curve, and draw LQP perpendicular to OY, also draw the ordinate MP, and the line FP. The line FP is called the **focal radius** of P.

Then $$ZO = OF = p,$$

and the equation of the directrix is $x + p = 0$, . . . (1)

while the focus is the point $(p, 0)$. . . . (2)

Again, from the definition of the parabola,

$$FP = LP; \text{ [geometric equation]}$$

but $FP = \sqrt{(x-p)^2 + y^2}$, and $LP = ZO + OM = p + x$;

hence $$\sqrt{(x-p)^2 + y^2} = (x + p),$$

whence $$\boldsymbol{y^2 = 4px}, \quad . \quad . \quad . \qquad [41]$$

which is the desired equation.

This first standard form [41] is the simplest equation of the parabola, and the one which will be most used in the subsequent study of the curve. It will be seen later (Chapter XII) that any equation which represents a parabola can be reduced to this form.

104. To trace the parabola $y^2 = 4px$. From equation [41] it follows:

(1) That the parabola passes through the point O, half way from the directrix to the focus. This point is called the **vertex** of the curve.

(2) That the parabola is symmetrical with regard to the

x-axis; *i.e.*, with regard to the line through the focus perpendicular to the directrix; this line is called the **axis*** of the curve.

(3) That x has always the same sign as the constant p, *i.e.*, that the entire curve and its focus lie on the same side of a line parallel to the directrix, and midway between the directrix and the focus.

(4) That x may vary in magnitude from 0 to ∞, and when x increases, so also does y (numerically); hence the parabola is an open curve, receding indefinitely from its directrix and its axis.

The parabola is then an open curve of one branch which lies on the same side of the directrix as does the focus; when constructed it has the form shown in Fig. 76.

105. Latus rectum. The chord through the focus of a conic, parallel to the directrix, is called its **latus rectum.** In the figure this chord is $R'R$.

Now $$R'R = 2\,FR = 2\,SR = 2\,ZF = 4\,p.$$

Hence *the length of the latus rectum of the parabola is* $4\,p$*;* that is, it is *equal to the coefficient of* x *in the first standard equation.*

106. Geometric property of the parabola. Second standard equation. Equation [41] may be interpreted as stating an intrinsic property of the parabola, — a property which belongs to every point of the parabola, whatever coördinate axes be chosen. For (see Fig. 76) the equation $y^2 = 4\,px$ gives the geometric relation

$$\overline{MP}^2 = 4\,OF \cdot OM = R'R \cdot OM,$$

or, expressed in words,

---

* The *axis of a curve* should be carefully distinguished from an *axis of coördinates;* though they often are coincident lines in the figures to be studied.

If from any point on the parabola, a perpendicular is drawn to the axis of the curve, the square on this perpendicular is equivalent to the rectangle formed by the latus rectum and the line from the vertex to the foot of the perpendicular.

This geometric property enables one to write down immediately the equation of the parabola, whenever the axis of the curve is parallel to one of the coördinate axes.

E.g., if the vertex of the parabola is the point $A \equiv (h, k)$, and its axis is parallel to the x-axis, as in the figure, let F be the focus and $P \equiv (x, y)$ be any point on the parabola; draw MP perpendicular to the axis AK. Then

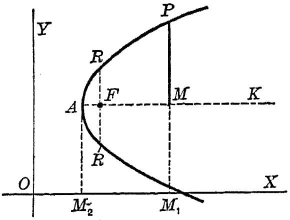


FIG. 77.

$$\overline{MP}^2 = 4\,AF \cdot AM,$$

i.e., $$(y-k)^2 = 4p(x-h), \quad [42]$$

which is the equivalent algebraic equation. This may be taken as a *second standard form* of the equation, representing the parabola with vertex at the point (h, k), with axis parallel to the x-axis, and, if p is positive, lying wholly on the positive side of the line $x = h$.

Equation [42] evidently may be reduced to equation [41] by a transformation of coördinates to parallel axes through the vertex (h, k), as the new origin.

Again, suppose the position of the parabola to be that represented in Fig. 78. The vertex is $A \equiv (h, k)$, and the axis of the parabola is parallel to the y-axis. Let $P \equiv (x, y)$ be any point on the curve, and draw MP perpendicular to the axis of the curve.

Then $$\overline{MP}^2 = 4\,AF \cdot AM \qquad \text{[geometric property]}$$
$$= 4\,(-p)AM, \quad \text{[since } AF \text{ is negative]}$$

whence, substituting the coördinates of A and P,

$$(x-h)^2=-4p(y-k), \quad . \quad . \quad . \quad [43]$$

which is another form for the second standard equation of the parabola.

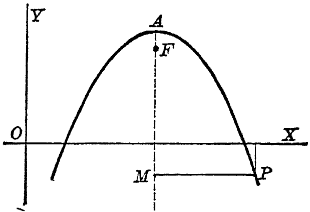


FIG. 78.

EXERCISES

Construct the following parabolas, and find their equations:

1. having the focus at the point $(^-1, 3)$, and for directrix the line $3x-5y=2$ (cf. Art. 102);

2. having the focus at the origin, and for directrix the line

$$2x-y+3=0;$$

3. with the vertex at the origin, and the focus at the point $(3, 0)$;

4. with the vertex at the origin, and the focus at the point $(0, ^-3)$;

5. with the vertex at the point $(^-2, 5)$, and the focus at the point $(^-2, 1)$;

6. with the vertex at the point $(^-2, ^-4)$, and the focus at the point $(1, ^-4)$;

7. having the focus at the point $(2p, 0)$, and for directrix the line $x=0$.

8. What is the latus rectum of each of the parabolas of exercises 3 to 6.

9. Describe the effect produced on the form of a parabola by increasing or decreasing the length of its latus rectum.

107. Every equation of the form $Ax^2+2Gx+2Fy+C=0$, or $By^2+2Gx+2Fy+C=0$, represents a parabola whose axis is parallel to one of the coördinate axes.

Equations [41], [42], and [43] are of the form

$$By^2+2Gx+2Fy+C=0, \text{ or } Ax^2+2Gx+2Fy+C=0;$$

that is, each has one and only one term containing the square of a variable, and no term containing the product of the two variables. Conversely, it may be shown that an equation of either of these forms represents a parabola whose axis is parallel to one of the coördinate axes.

A numerical example will first be discussed, by the method which has already been employed in connection with the equation of the circle (Art. 79), and which is applicable also in the case of the other conics. It is the method of reducing the given equation to a standard form, and is analogous to "completing the square" in the solution of quadratic equations.

Example. Given the equation

$$25y^2 - 30y - 50x + 89 = 0,$$

to show that it represents a parabola; and to find its vertex, focus, and directrix.

Divide both members of the equation by 25, and complete the square of the y-terms; the equation may then be written

$$y^2 - \tfrac{6}{5}y + \tfrac{9}{25} = 2x - \tfrac{89}{25} + \tfrac{9}{25},$$

that is,

$$(y - \tfrac{3}{5})^2 = 2(x - \tfrac{8}{5}),$$

whence

$$(y - \tfrac{3}{5})^2 = 4 \cdot \tfrac{1}{2} \cdot (x - \tfrac{8}{5}).$$

Now this equation is in the second standard form (cf. equation [42]), and therefore every point on its locus has the geometric property given in Art. 106; and the locus is a parabola. The vertex is at the point $(\frac{8}{5}, \frac{3}{5})$; its axis is parallel to the x-axis, extending in the positive direction; and, since $p = \frac{1}{2}$, its focus is at the point $(\frac{21}{10}, \frac{3}{5})$, and the directrix is the line $x = \frac{11}{10}$.

Consider now the general equation, and apply the same method, taking for example the second form, viz.:

$$Ax^2 + 2Gx + 2Fy + C = 0.$$

Dividing both numbers of the equation by A, completing the square of the x-terms, and transposing, the equation becomes

$$x^2 + 2\frac{G}{A}x + \frac{G^2}{A^2} = -2\frac{F}{A}y - \frac{C}{A} + \frac{G^2}{A^2},$$

that is, $$\left(x+\frac{G}{A}\right)^2=-\frac{2F}{A}\left(y-\frac{G^2-AC}{2AF}\right),$$

whence $$\left(x+\frac{G}{A}\right)^2=4\left(-\frac{F}{2A}\right)\left(y-\frac{G^2-AC}{2AF}\right).$$

Comparing this equation with the standard equation [43], it is seen that its locus is a parabola, whose axis is parallel to the y-axis, extending in the negative direction if A and F have like signs, and in the positive direction if A and F have unlike signs. Its vertex is at the point $\left(-\frac{G}{A}, \frac{G^2-AC}{2AF}\right)$; and, since $p=-\frac{F}{2A}$, its focus is at the point

$$\left(-\frac{G}{A}, \frac{G^2-F^2-AC}{2AF}\right),$$

and its directrix is the line $y=\frac{G^2+F^2-AC}{2AF}$.

NOTE. The transformation just given fails if $A=0$ or if $F=0$, for in that case some of the terms in the last equation are infinite. If, however, $A=0$, the given equation becomes $2Gx+2Fy+e=0$; and, this being of the first degree, represents a straight line. If, on the other hand, $F=0$, the given equation reduces to $Ax^2+Gx+C=0$, and represents two straight lines each parallel to the y-axis; they are real and distinct, real and coincident, or imaginary, depending upon the value of G^2-AC. These lines may be regarded as limiting forms of the parabola (see Chapter XII).

EXERCISES

Determine the vertex, focus, latus rectum, equation of the directrix and of the axis for each of the following parabolas; also sketch each of the figures:

1. $y^2-5x+4y-10=0$;

2. $3x^2+12x+4y-8=0$;

3. $5y-1=3y^2+4x$;

4. $y^2+2y-12x-11=0$.

108. Reduction of the equation of a parabola to a standard form. In Art. 102 it was shown that the equation of a parabola having any given directrix and focus is in general not as simple as the standard equation. It will now be shown that if the coördinate axes be transformed so as to be parallel to the axis and directrix of the curve, the equation will be reduced to a standard form. For example, the equation of the parabola with focus at $(2, -3)$, and having for directrix the line $x-2y-1=0$, was found to be

$$4x^2+4xy+y^2-18x+26y+64=0.$$

The axis of the curve is a line through $(2, -3)$ and perpendicular to

$$x-2y-1=0;$$

its equation is $2x+y=1$, and it cuts the x-axis at the angle $\theta=\tan^{-1}(-2)$.

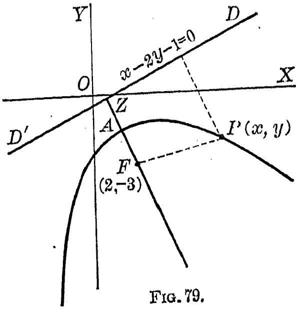


FIG. 79.

The point Z is the intersection of the directrix and axis, and may be found from the two linear equations representing these lines; the vertex A is the point bisecting ZF. If, then, the axes are rotated through the angle $\theta=\tan^{-1}(-2)$, the equation will be reduced to the second standard form, [42]; and if the origin be also removed to the vertex A, the equation will be further reduced to the first standard form, [41]. The point Z is $(\frac{3}{5}, -\frac{1}{5})$, A is $(\frac{13}{10}, -\frac{8}{5})$; hence, $p = AF = \dfrac{7}{\sqrt{5}}$, and transforming the axes through the angle $\theta=\tan^{-1}(-2)$, to the new origin $(\frac{13}{10}, -\frac{8}{5})$, the equation of the parabola reduces to $y^2=\dfrac{28}{\sqrt{5}}x$.

The problem of reducing any equation representing a parabola to its standard form is taken up more fully in Chap. XII.

EXERCISES

Find, and reduce to the first standard form, the equation of each of the following parabolas; also make a sketch of each figure:

1. with focus at the point $(-1, 3)$, and having for directrix the line

$$3x-5y=2;$$

2. with focus at the point $(-8, -\frac{1}{2})$, and having for directrix the line

$$2x+7y-8=0;$$

3. with focus at the point (a, b), and having for directrix the line

$$\frac{x}{a}+\frac{y}{b}=1.$$

II. THE ELLIPSE

Special Equation of the Second Degree

$$Ax^2 + By^2 + 2\,Gx + 2\,Fy + C = 0$$

109. The ellipse defined. An ellipse is the locus of a point which moves so that the ratio of its distance from a fixed point, called the **focus,** to its distance from a fixed line, called the **directrix,** is constant and less than unity. The constant ratio is called the **eccentricity** of the ellipse. This curve is the conic section with eccentricity $e < 1$. (cf. Art. 48.)

The equation of an ellipse with any given focus, directrix, and eccentricity may be readily obtained from this definition.

EXAMPLE. An ellipse of eccentricity $\frac{2}{3}$ has its focus at $(2, {}^{-}1)$, and has the line $x + 2\,y = 5$ for directrix. Let $P \equiv (x, y)$ (Fig. 85) be any point on the curve, F the focus, and PQ the perpendicular from P to the directrix.

Then
$$FP = \tfrac{2}{3}\,QP;$$
but $FP = \sqrt{(x-2)^2 + (y+1)^2}$, $QP = \dfrac{x + 2\,y - 5}{+\sqrt{5}}$ (Arts. 26, 64),

hence
$$(x-2)^2 + (y+1)^2 = \tfrac{4}{45}\,(x + 2\,y - 5)^2;$$
that is,
$$41\,x^2 - 16\,xy + 29\,y^2 - 140\,x + 170\,y + 125 = 0;$$
which is the equation of the given ellipse.

As in the case of the parabola, so also here, a particular choice of the coördinate axes gives a simpler form for the equation of the ellipse; an equation which is more suitable for the study of the curve, and to which every equation representing an ellipse can be reduced. As has been seen in Art. 48, the curve is symmetrical with respect to the line through the focus and perpendicular to the directrix; and cuts that line in two points, one on either side of the focus. The equation of the ellipse will be in a simpler form if this line of symmetry is chosen as the x-axis, with the origin half way between its two points of intersection with the curve. The resulting equation is the *first standard form* of the equation of the ellipse.

110. The first standard equation of the ellipse. Let F be the focus, $D'D$ the directrix, and ZFX the perpendicular to DD' through F, cutting the curve in the two points A' and A (Art. 48)*. Denote by $2a$ the length of AA', and let O be its middle point, so that

$$A'O = OA' = a.$$

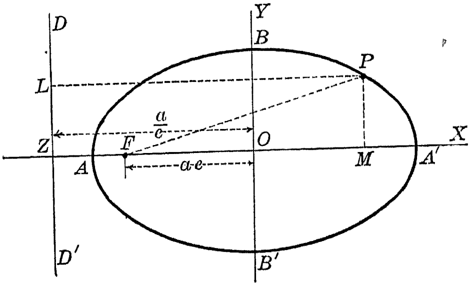


Fig. 80.

Let ZX be the x-axis, O the origin, and OY, perpendicular to OX, the y-axis. Then, by the definition of the ellipse,

$$AF = eZA, \quad \text{and} \quad FA' = eZA';$$

$\therefore \quad AF + FA' = e(ZA + ZA') = e(ZA + ZA + AA'),$

i.e., $\quad AA' = e(2\,ZA + AA'),$

whence $\quad 2\,a = 2e(ZA + AO) = 2\,eZO;$

therefore $\quad ZO = \dfrac{a}{e},$

and the *equation of the directrix is* $x + \dfrac{a}{e} = 0$. . . . . (1)

Again, $\quad FA' - AF = e(ZA' - ZA);$

i.e., $\quad FO + OA' - (AO - FO) = eAA',$

whence $\quad 2\,FO = 2\,ae;$

---

* This equation may also be easily derived independently of Art. 48, — cf. Arts. 103, 116.

therefore $$FO = ae,$$

and the *focus F is the point* $(-ae, 0)$. . . . . (2)

Now, for any point P on the curve, draw the ordinate MP and the perpendicular LP to the directrix; then

$$FP = eLP, \quad \text{[geometric equation]} \quad . \quad . \quad . \qquad (3)$$

but $FP = \sqrt{(x+ae)^2 + y^2}$, $LP = \frac{a}{e} + x$;

hence $$(ae+x)^2 + y^2 = e^2\left(x + \frac{a}{e}\right)^2, \quad . \quad . \quad . \quad . \qquad (4)$$

that is, $$(1-e^2)x^2 + y^2 = a^2(1-e^2), \quad . \quad . \quad . \qquad (5)$$

that is, $$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

From equation (6), the intercepts of the curve on the y-axis are $\pm a\sqrt{1-e^2}$. Both intercepts are real, since $e < 1$; hence the ellipse cuts the y-axis in two real points, B and B', on opposite sides of the origin O and equidistant from it. If OB is denoted by $+b$, so that

$$b^2 = a^2(1-e^2), \quad . \quad . \quad . \qquad (7)$$

equation (3) takes the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.* \quad . \quad . \quad . \qquad [44]$$

This is the simplest equation of the ellipse, and will be most used in the subsequent study of the properties of that curve. As will be seen in Chapter XII, every equation representing an ellipse can be reduced to this form.

---

* If $a = b$ (*i.e.*, if $e = 0$) this equation represents a circle. The ellipse, then, includes the circle as a special case. In other words: a circle is an ellipse whose eccentricity is zero.

111. To trace the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. From equation [44] it follows that:

(1) The ellipse is symmetrical with regard to the x-axis; *i.e.*, with regard to the line through the focus and perpendicular to the directrix; this line is therefore called the **principal axis** of the curve;

(2) The ellipse is symmetrical with regard to the y-axis also; *i.e.*, with regard to a line parallel to the directrix and passing through the mid-point of the segment AA' (Fig. 81) which the curve cuts from its principal axis;

(3) For every value of x from $-a$ to $+a$, the two corresponding values of y are real, equal numerically, but opposite in sign; and for every value of y from $-b$ to b, the two values of x are real and equal numerically, but opposite in sign; and that neither x nor y can have real values beyond these limits.

The ellipse is, therefore, a closed curve, of one branch, which lies wholly on the same side of the directrix as the focus; and the curve has the form represented in Fig. 80, — which agrees with the foot-note on p. 71.

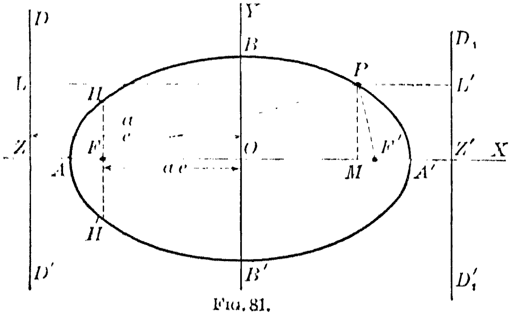


Fig. 81.

The segment AA' (Fig. 81) of the principal axis intercepted by the curve is called its **major** or **transverse axis;**

the corresponding segment $B'B$ is its **minor** or **conjugate axis.** From the symmetry of the curve with respect to these axes it follows that it is also symmetrical with respect to their intersection O, the **center** of the ellipse. It follows also that the ellipse has a second focus at $F' \equiv (ae, 0)$ (Fig. 81) and a second directrix D'_1D_1—the line $x - \frac{a}{e} = 0$—on the positive side of the minor axis, and symmetrical to the original focus and directrix, respectively.*

The **latus rectum** of the ellipse, *i.e.*, the focal chord parallel to the directrix (Art. 105), is evidently twice the ordinate of the point whose abscissa is ae.

But if $x_1 = ae$, $y_1 = b\sqrt{1 - e^2}$; or, since $b = a\sqrt{1 - e^2}$, $y_1 = \frac{b^2}{a}$. Hence the latus rectum is $\frac{2\,b^2}{a}$.

112. Intrinsic property of the ellipse. Second standard equation. Equation [44] states a geometric property which belongs to every point of the ellipse, whatever the coördinate axes chosen, and to no other point: viz., if P be any point of the ellipse (Fig. 80), then

$$\frac{\overline{OM}^2}{\overline{OA'}^2} + \frac{\overline{MP}^2}{\overline{OB}^2} = 1;$$

that is, in words :

---

* To show this analytically, let $OF' = ae$, and $OZ' = \frac{a}{e}$, and let $P \equiv (x, y)$ be any point on the ellipse, as before. Equation (5), of Art. 110, gives the relation between x and y; expanding equation (5), and subtracting $4\,aex$ from each member, it becomes

$$a^2e^2 - 2\,aex + x^2 + y^2 = a^2 - 2\,aex + e^2x^2,$$

which may be written

$$(ae - x)^2 + y^2 = e^2\left(\frac{a}{e} - x\right)^2.$$

i.e.,
$$\overline{F'P}^2 = e^2\,\overline{PL'}^2;$$

which shows that P is on an ellipse whose focus is F' and whose directrix is D'_1D_1.

If from any point on the ellipse a perpendicular be drawn to the transverse axis; then the square of the distance from the center of the ellipse to the foot of this perpendicular, divided by the square of the semi-transverse axis, plus the square of the perpendicular divided by the square of the semi-conjugate axis, equals unity.

This geometric or physical property belongs to no point not on the curve, and therefore completely determines the ellipse. It enables one to write immediately the equation of any ellipse whose axes are parallel to the coördinate axes.

For example: if, as in Fig. 82, the major axis of an ellipse is parallel to the x-axis, and the center is at the point

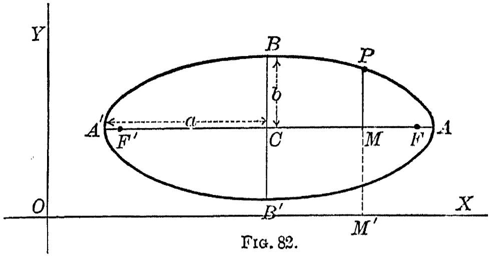


Fig. 82.

$C \equiv (h, k)$, let $P \equiv (x, y)$ be any point on the curve, and a, b be the semi-axes, then

$$\frac{\overline{CM}^2}{\overline{CA}^2} + \frac{\overline{MP}^2}{\overline{CB}^2} = 1,$$

that is $$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, \quad . \quad . \quad . \quad [45]$$

which is the equation of the given ellipse.

Or again, if, as in Fig. 83, the major axis is parallel to the y-axis; then, as before

$$\frac{\overline{CM}^2}{\overline{CA}^2} + \frac{\overline{MP}^2}{\overline{CB}^2} = 1,$$

i.e., $$\frac{(y-k)^2}{a^2}+\frac{(x-h)^2}{b^2}=1, \quad . \quad [46]$$

which is the equation of the given ellipse.

Equation [45] may be considered a *second standard* form of the equation of the ellipse; by a change of coördinates to a set of parallel axes through the center $C \equiv (h, k)$, as the new origin, it can be reduced to the first standard form.

By Art. 110 the distance from the center of an ellipse to its focus is ae; but since $b^2 = a^2(1-e^2)$* [Art. 110, eq. (7)], therefore $ae = \sqrt{a^2-b^2}$; hence, in Figs. 82 and 83,

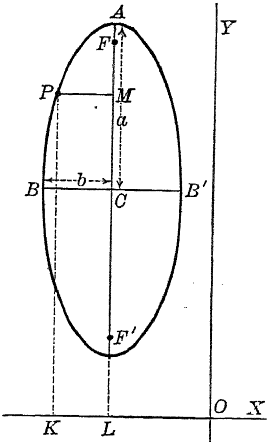


FIG. 83.

$$F'C = CF = ae = \sqrt{a^2-b^2}.$$

Again, the equation of an ellipse, in either standard form, gives the semi-axes as well as the center of the curve, therefore the positions of the foci are readily determined from either standard form of the equation.

EXERCISES

Construct the following ellipses, and find their equations:

1. given the focus at the point $(-1, 1)$, the equation of the directrix $x - y + 3 = 0$, and the eccentricity $\frac{1}{2}$ (cf. Art. 109);

2. given the focus at the origin, the equation of the directrix $x = -9$, and the eccentricity $\frac{1}{3}$;

---

* The student should observe that b is the *semi-minor-axis* and not necessarily the denominator of y^2 in the standard forms of the equation of the ellipse — [44], [45], or [46]; he should also observe that the foci are always on the *major axis*.

3. given the focus at the point (0, 1), the equation of the directrix $y - 25 = 0$, and the eccentricity $\frac{1}{5}$;

4. given the center at the origin, and the semi-axes $\sqrt{2}$, $\sqrt{5}$. Find also the latus rectum.

Find the equation of an ellipse referred to its center, whose axes are the coördinate axes, and

5. which passes through the two points (2, 2) and (3, 1).

6. whose foci are the points (3, 0), ($^-3$, 0), and eccentricity $\frac{1}{3}$.

7. whose foci are the points (0, 6), (0, $^-6$), and eccentricity $\frac{2}{3}$.

8. whose latus rectum is 5, and eccentricity $\frac{2}{3}$.

9. whose latus rectum is 8, and the major axis 10.

10. whose major axis is 18, and which passes through the point 6, 4.

Draw the following ellipses, locate their foci, and find their equations:

11. given the center at the point (3, $^-2$), the semi-axes 4 and 3, and the major axis parallel to the x-axis (cf. Art. 112);

12. given the center at the point ($^-8$, 1), the semi-axes 2 and 5, and the major axis parallel to the y-axis;

13. given the center at the point (0, 7), the origin at a vertex, and (2, 3) a point on the curve;

14. given the circumscribing rectangle, whose sides are the lines $x + 1 = 0$, $2x - 3 = 0$, $y + 6 = 0$, $3y + 4 = 0$; the axes of the curve being parallel to the coördinate axes.

15. If b becomes more and more nearly equal to a, what curve does the ellipse approach as a limit?

113. Every equation of the form $Ax^2 + By^2 + 2Gx + 2Fy + C = 0$, in which A and B have the same sign, represents an ellipse whose axes are parallel to the coördinate axes. Equations [44], [45], and [46], obtained for the ellipse, are all, when expanded, of the form

$$Ax^2 + By^2 + 2Gx + 2Fy + C = 0, \quad . \quad . \quad . \quad (1)$$

where *A and B have the same sign, and neither of them is zero.* Conversely, an equation of this form represents an ellipse whose axes are parallel to the coördinate axes. As in Art. 107, a numerical case will first be examined, and then the general equation taken up in a similar manner.

Example. Given the equation $4x^2 + 9y^2 - 16x + 18y - 11 = 0$, to show that it represents an ellipse, and to find its elements. Completing

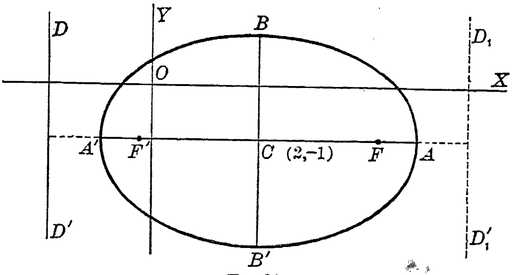


Fig. 84.

the square for the terms in x, and also for those in y, and transposing, this equation becomes

$$4x^2 - 16x + 16 + 9y^2 + 18y + 9 = 11 + 16 + 9,$$

that is,
$$4(x-2)^2 + 9(y+1)^2 = 36;$$

hence
$$\frac{(x-2)^2}{3^2} + \frac{(y+1)^2}{2^2} = 1.$$

This equation is of the form [45], and, therefore, its locus has the geometric property given in Art. 112, and is an ellipse. Its center is the point $(2, -1)$; its major axis is parallel to the x-axis, of length 6; its minor axis is of length 4; the foci are the points

$$F' = (2-\sqrt{5}, -1),\ F = (2+\sqrt{5}, -1);$$

and the equations of the directrices are, respectively,

$$x = 2 + \frac{9}{\sqrt{5}}, \qquad x = 2 - \frac{9}{\sqrt{5}}.$$

Following the method illustrated above, of completing the squares, the general equation (1) may be written

$$A\left(x^2 + 2\frac{G}{A}x + \frac{G^2}{A^2}\right) + B\left(y^2 + 2\frac{F}{B}y + \frac{F^2}{B^2}\right) = -C + \frac{G^2}{A} + \frac{F^2}{B}.$$

that is,

$$A\left(x+\frac{G}{A}\right)^2+B\left(y+\frac{F}{B}\right)^2=\frac{BG^2+AF^2-ABC}{AB};$$

which becomes, if the second member be represented by K,

$$\frac{\left(x+\frac{G}{A}\right)^2}{\frac{K}{A}}+\frac{\left(y+\frac{F}{B}\right)^2}{\frac{K}{B}}=1. \quad . \quad . \quad . \quad (2)$$

Comparing this equation with [45] or [46], it is seen to express the geometric relation of Art. 112, and therefore represents an ellipse. Its axes are parallel to the coördinate axes, its center is at the point $\left(-\frac{G}{A}, -\frac{F}{B}\right)$, and the lengths of the semi-axes are

$$a=\sqrt{\frac{K}{A}},\ b=\sqrt{\frac{K}{B}}.$$

The foci and directrices may be found as above.

Note. If $A = B$, then equation (1) represents a circle (Art. 79). If $ABC > BG^2 + AF^2$, equation (1) having been written with A and B positive, then no *real* values of x and y can satisfy equation (2), which is only another form of equation (1), and it is said to represent an *imaginary ellipse*. If $ABC = BG^2 + AF^2$, then $x = -\frac{G}{A}$, and $y = -\frac{F}{B}$ are the only real values that satisfy equation (2); in that case, this equation is said to represent a *point ellipse;* or, from another point of view, two imaginary lines which intersect in the real point $\left(-\frac{G}{A}, -\frac{F}{B}\right)$. Each of the above may be regarded as a limiting form of the ellipse.

EXERCISES

Determine, for each of the following ellipses, the center, semi-axes, foci, vertices, and latus rectum; then sketch each curve.

1. $3x^2 + 9y^2 - 6x - 27y + 2 = 0.$

2. $4x^2 + y^2 - 8x + 2y + 1 = 0.$

3. $x^2 + 15y^2 + 4x + 60y + 15 = 0.$

4. By completing the squares of the x-terms and of the y-terms, and a suitable transformation of coördinates, reduce the equations of exercises 1, 2, and 3 to the standard form [44].

114. Reduction of the equation of an ellipse to a standard form.

It is now evident that, if the directrix and focus of an ellipse are known, as in the example of Art. 109, the transformation of coördinates which is necessary to reduce the equation to a standard form can easily be determined. To illustrate: the ellipse of eccentricity $\frac{2}{3}$, with focus at $F \equiv (2, -1)$, and having for directrix the line $D'D$, whose equation is $x + 2y = 5$, has for its equation (Art. 109)

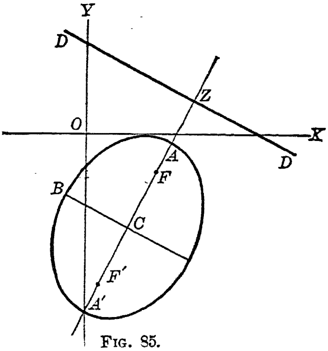


FIG. 85.

$$41x^2 - 16xy + 29y^2 - 140x + 170y + 125 = 0.$$

Its axis FZ, perpendicular to $D'D$, has the equation $2x - y = 5$, and cuts the x-axis at the angle $\tan^{-1}2$. If then the coördinate axes are rotated through the angle $\tan^{-1}2$, the equation will be reduced to the second standard form. Again, Z may be found as the intersection of the directrix and axis; it is the point $(3, 1)$. Then A and A', the vertices of the ellipse, divide FZ internally and externally in the ratio $\frac{2}{3}$; hence (Art. 30) these coördinates are $(\frac{12}{5}, {}^-\frac{1}{5})$, $(0, {}^-5)$. Also C, the center of the ellipse, is the point $(\frac{6}{5}, {}^-\frac{13}{5})$. If the origin be next transformed to the point C, the equation will be reduced to the first standard form. Since the axis AA' is of length $\dfrac{12}{\sqrt{5}}$, and the eccentricity is $\frac{2}{3}$, the semi-axes are $\dfrac{6}{\sqrt{5}}$ and 2; hence the reduced equation, with C as origin and CA as x-axis, will be

$$\frac{x^2}{\frac{36}{5}} + \frac{y^2}{4} = 1.$$

The problem of reducing to standard form the equation of an ellipse, when the directrix is not known, will be postponed to Chapter XII.

EXERCISES

Find, and reduce to the first standard form, the equation of the ellipse:

1. with focus at the point $(1, {}^-3)$, with the line $x + y = 7$ for directrix, and eccentricity $\frac{1}{2}$;

2. with focus at the point (a, b), the line $\dfrac{x}{a} + \dfrac{y}{b} = 1$ for directrix, and eccentricity $\dfrac{l}{n}$ (where $l < n$).

III. THE HYPERBOLA

Special Equation of the Second Degree

$$\boldsymbol{Ax^2 - By^2 + 2Gx + 2Fy + C = 0}$$

115. The hyperbola defined. An hyperbola is the locus of a point which moves so that the ratio of its distance from a fixed point, called the **focus,** to its distance from a fixed line, called the **directrix,** is constant and greater than unity. The constant ratio is the **eccentricity** of the hyperbola. This curve is the conic section with eccentricity $e > 1$ (cf. Art. 48).

Since the hyperbola differs from the ellipse only in the sign of $1 - e^2$, which is $+$ in the ellipse and $-$ in the hyperbola, the standard equation of the hyperbola can be derived by the method of Art. 110; and it will be found that with choice of axes and notation as there given, the results given in eqs. (1), (2), and (3) of that article apply equally to the hyperbola. If now, since $1 - e^2$ is negative, the substitution $b^2 = a^2(e^2 - 1)$ is made, equation (6) (p. 181) will become

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad . \quad . \quad . \quad [47]$$

which is the simplest equation of the hyperbola. For variety, this equation will be obtained by a different method.

116. The first standard form of the equation of the hyperbola. Let F be the focus, $D'D$ the directrix, and e the eccentricity of the curve. Take $D'D$ as the y-axis, with the perpendicular OFX upon it, through the focus, as the x-axis. Let $2p$ denote the given distance OF, and let

$$P \equiv (x, y)$$

be any point of the locus, with coördinates LP and MP.

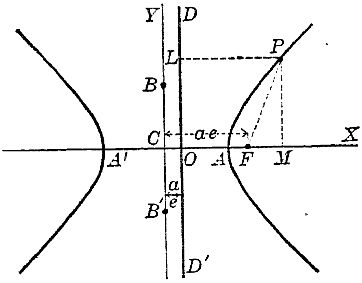


FIG. 86.

Then $\qquad FP = eLP; \qquad$ [geometric equation]

but $\qquad FP = \sqrt{(x - 2p)^2 + y^2}$, and $MP = x$;

$$\therefore \qquad (x - 2p)^2 + y^2 = e^2x^2,$$

that is, $\qquad (e^2 - 1)x^2 + y^2 + 4px - 4p^2 = 0, \quad . \quad . \quad . \quad (1)$

which is the equation of the hyperbola referred to its directrix and principal axis as coördinate axes (cf. Art. 48).

The curve cuts the x-axis in two points, $A \equiv (x_1, 0)$, and $A' \equiv (x_2, 0)$, — the **vertices** of the hyperbola, — whose abscissas are determined by the equation

$$(e^2 - 1)x^2 + 4px + 4p^2 = 0.$$

The abscissa of C, the middle point of the segment AA', is, therefore,

$$OC = \frac{x_1 + x_2}{2} = \frac{-2p}{e^2 - 1} \quad \text{(Art. 11)};$$

hence the center is on the opposite side of the directrix from the focus.

Now transform equation (1) to a parallel set of axes through C; the equations for transformation are (Art. 71)

$$x = x' - \frac{2p}{e^2 - 1}, \text{ and } y = y';$$

substituting these values, and removing accents, eq. (1) becomes

$$(e^2 - 1)\left(x - \frac{2p}{e^2 - 1}\right)^2 + y^2 + 4p\left(x - \frac{2p}{e^2 - 1}\right) - 4p^2 = 0$$

which reduces to $(e^2 - 1)x^2 + y^2 = \dfrac{4p^2e^2}{e^2 - 1}$,

that is,

$$\frac{x^2}{\dfrac{4p^2e^2}{(e^2 - 1)^2}} - \frac{y^2}{\dfrac{4p^2e^2}{e^2 - 1}} = 1. \quad . \quad . \quad . \quad (2)$$

If these denominators are represented by a^2 and b^2 respectively, *i.e.*, if

$$a^2 = \frac{4p^2e^2}{(e^2 - 1)^2}, \text{ and } b^2 = \frac{4p^2e^2}{e^2 - 1}, \quad . \quad . \quad . \quad (3)$$

then

$$b^2 = a^2(e^2 - 1), \quad . \quad . \quad . \quad (4)$$

and equation (2) may be written in the simple form

$$\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}, \quad . \quad . \quad . \quad [47]$$

the standard equation of the hyperbola. Every equation representing an hyperbola can be reduced to this form, as is shown in Chapter XII.

The distance from the center to the focus of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is easily found thus:

$$CF = CO + OF$$

$$= \frac{2p}{e^2-1}+2p=\frac{2pe^2}{e^2-1};$$

but, from equation (2),

$$a=\frac{2pe}{e^2-1},$$

hence $CF = ae$,

therefore *the focus F is the point* $(ae, 0)$. . . . (4)

Similarly for the directrix:

$$CO=\frac{2p}{e^2-1}=\frac{a}{e},$$

hence *the directrix is the line* $x-\frac{a}{e}=0$. . . . (5)

As above defined, b is real, and its value is known when a and e are known. In Fig. 86,

$$OB=b,\ OB'=-b,\ \text{and}\ \ b=a\sqrt{e^2-1}.$$

117. To trace the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Equation [47] shows that:

(1) The hyperbola is symmetrical with regard to the x-axis; that is, with respect to the line through the focus and perpendicular to the directrix. This line is therefore called the **principal axis** of the hyperbola;

(2) The hyperbola is symmetrical with regard to the y-axis also; *i.e.*, with regard to the line parallel to the directrix and passing through the mid-point of the segment cut by the curve from its principal axis;

(3) For every value of x from $-a$ to a, y is imaginary; while for every other value of x, y is real and has two values, equal numerically but opposite in sign. But for every value of y, x has two real values, equal numerically and opposite in sign. When x increases numerically from a to ∞, then y increases also numerically from 0 to ∞.

These facts show that no part of the hyperbola lies between the two lines perpendicular to its principal axis and drawn through the vertices of the curve; but that it has two open infinite branches, lying outside of these two lines. The form of the hyperbola is as represented in Fig. 86.

The segment $A'A$ of the principal axis, intercepted by the curve, is called its **transverse axis.** The segment $B'B$ of the second line of symmetry (the y-axis), where $B'O = OB = b$, is called the **conjugate axis;** and although not cut by the hyperbola, it bears important relations to the curve. From the symmetry of the hyperbola, with respect to these axes, it follows that it is also symmetrical with respect to their intersection O, the **center** of the curve. It follows also that there is a second focus at the point $(-ae, 0)$, and a second directrix in the line $x + \frac{a}{e} = 0$ on the negative side of the conjugate axis, and symmetrical to the original focus and directrix. (See Art. 111, foot-note.)

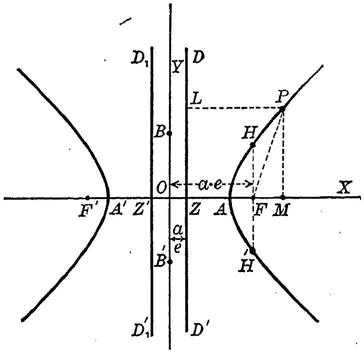


FIG. 87.

The **latus rectum** of the hyperbola is readily found to be $\frac{2b^2}{a}$ (cf. Arts. 105, 111).

118. Intrinsic property of the hyperbola. Second standard equation. Equation [47] states a geometric property which belongs to every point of an hyperbola, whatever the coördinate axes chosen, and to no other point; and which therefore completely defines the hyperbola. With the figure and notation of Art. 117, equation [47] states (Fig. 87)

$$\frac{\overline{OM}^2}{\overline{OA}^2} - \frac{\overline{MP}^2}{\overline{OB}^2} = 1,$$

a property entirely analogous to that of Art. 112 for the ellipse. It enables one to write at once the equation of an hyperbola with given center and semi-axes, and axes parallel to the coördinate axes.

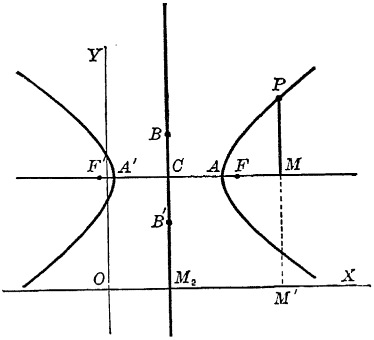


Fig. 88.

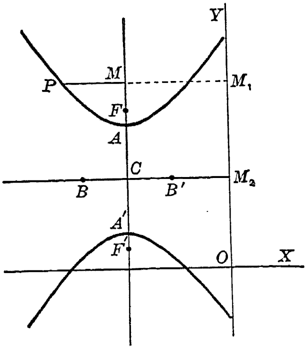


Fig. 89.

For example, if the transverse axis is parallel to the x-axis, as in Fig. 88, and the center at the point $C \equiv (h, k)$, and if $P \equiv (x, y)$ is any point on the curve; then

$$\frac{\overline{CM}^2}{\overline{CA}^2} - \frac{\overline{MP}^2}{\overline{CB}^2} = 1,$$

$$i.e., \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1, \qquad [48]$$

which is the equation of the hyperbola, with a and b as semi-axes.

Again, if the transverse axis is parallel to the y-axis, as in Fig. 89, with the center at the point (h, k), the equation of the hyperbola will be found to be

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1,$$

i.e.,

$$\frac{(x-h)^2}{b^2} - \frac{(y-k)^2}{a^2} = -1. \quad . \quad . \quad . \qquad [49]$$

NOTE 1. That the expressions obtained on p. 193 for the distances from the center to the focus and the directrix, of hyperbola [47], are equally true for hyperbolas [48] and [49] follows from the fact that those expressions involve only a, b, and e; moreover, equation (4) of Art. 116 determines e in terms of a and b; hence, for all these hyperbolas, $e^2 = \frac{a^2+b^2}{a^2}$, the distances from the center to the foci are given by

$$CF = ae = \pm\sqrt{a^2+b^2},$$

and those to the directrices by

$$CZ = \frac{a}{e} = \frac{a^2}{\pm\sqrt{a^2+b^2}}.$$

NOTE 2. It should be noticed that in equations [47], [48], [49], the negative term involves that one of the coördinates which is parallel to the conjugate axis.

EXERCISES

1. Find the equation of the hyperbola having its focus at the point $(^-1, ^-1)$, for its directrix the line $3x - y = 7$, and eccentricity $\frac{4}{3}$. Plot the curve (cf. Art. 105, and Art. 109, Ex.).

Find the equation of the hyperbola whose center is at the origin and

2. whose semi-axes equal, respectively, 5 and 3 (cf. Art. 116, [47]);

3. with transverse axis 8, — the point (20, 5) being on the curve;

4. the distance between the foci 5, and eccentricity $\sqrt{2}$;

5. with the distance between the foci equal to twice the transverse axis.

Find the equation of an hyperbola

6. with center at the point $(3, ^-2)$, semi-axes 4 and 3, and the transverse axis parallel to the x-axis. Plot the curve (cf. Art. 118);

7. with center at the point $(^-3, {}^-4)$, semi-axes 6 and 2, and the transverse axis parallel to the y-axis. Plot the curve.

8. Find the foci and latus rectum for the hyperbolas of exercises 6 and 7.

9. By a suitable transformation of coördinates, reduce the equations of exercises 6 and 7 to the standard form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

10. Find the foci of the hyperbolas

$$(\alpha)\quad \frac{x^2}{25}-\frac{y^2}{9}=1,\qquad (\beta)\quad \frac{x^2}{4}-\frac{y^2}{9}=1,\qquad (\gamma)\quad \frac{y^2}{9}-\frac{x^2}{4}=1.$$

Plot the curves (β) and (γ).

119. Every equation of the form $Ax^2+By^2+2\,Gx+2\,Fy+C=0$, in which A and B have unlike signs, represents an hyperbola whose axes are parallel to the coördinate axes. When cleared of fractions and expanded, the three equations found for the hyperbola are of the form

$$Ax^2+By^2+2\,Gx+2\,Fy+C=0, \quad . \quad . \quad . \quad (1)$$

where A and B have opposite signs, and neither of them is zero. Conversely, it will now be shown that every equation of this form represents an hyperbola, whose axes are parallel to the coördinate axes. A numerical case will be examined first, and then the general equation.

EXAMPLE. To show that the equation $9\,x^2-4\,y^2-18\,x+24\,y-63=0$ represents an hyperbola, and to find its elements. Transposing the constant term, and completing the squares of the x-terms and y-terms, the equation may be written

$$9(x-1)^2-4\,(y-3)^2=36,$$

i.e.,

$$\frac{(x-1)^2}{2^2}-\frac{(y-3)^2}{3^2}=1.$$

Since this equation is of the form [48], its locus has the geometric property given in Art. 118, and therefore represents an hyperbola. Its center is at the point (1, 3), its transverse axis is parallel to the x-axis, of length 4, and its conjugate axis is of length 6. The eccentricity is $e=\frac{1}{2}\sqrt{13}$, the foci are at the points $(1-\sqrt{13}, 3)$ and $(1+\sqrt{13}, 3)$; and the directrices are the lines whose equations are

$$x = 1 \pm \frac{4}{\sqrt{13}}.$$

Following the method illustrated in the numerical example, the general equation (1) may be written in the form

$$\frac{\left(x+\frac{G}{A}\right)^2}{\frac{K}{A}} + \frac{\left(y+\frac{F}{B}\right)^2}{\frac{K}{B}} = 1, \quad . \quad . \quad . \quad (2)$$

wherein (cf. Art. 113, p. 188),

$$K = \frac{BG^2 + AF^2 - ABC}{AB}.$$

Since A and B have opposite signs, the two terms in the first member of this equation are of opposite signs; the equation is therefore in the form of [48] or [49], and represents an hyperbola. Its axes are parallel to the coördinate axes, its center is the point $\left(-\frac{G}{A}, -\frac{F}{B}\right)$, and its semi-axes are $\sqrt{\pm\frac{K}{A}}$* and $\sqrt{\pm\frac{K}{B}}$.

NOTE. Since A and B have opposite signs, equation (2), which is only another form of equation (1), always represents a real locus; it is an hyperbola proper except when $ABC = BG^2 + AF^2$, and it then represents a pair of intersecting straight lines (cf. Art. 67).

It is clear that the method shown for the ellipse in Art. 114 can be applied equally well to the hyperbola, to reduce any equation of this curve to the standard form, when the directrix is known. The problem of reducing to the standard form the general equation of an ellipse, when the directrix and focus are not known, is considered in full in Chapter XII.

---

* That sign (+ or −) which makes the fraction positive is to be used.

EXERCISES

Determine for each of the following hyperbolas the center, semi-axes, foci, vertices, and latus rectum:

1. $16x^2 - 8y^2 + 64x - 36y + 10 = 0$;

2. $x^2 - 5y^2 + 15y - 10x + 1 = 0$;

3. $2x + 6y + 3y^2 = x^2 + 7$.

4. Reduce the equations of exercises 1, 2, 3, to the standard form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Sketch each curve.

120. Summary. In the preceding articles it has been shown that the special equation of the second degree,

$$Ax^2 + By^2 + 2Gx + 2Fy + C = 0,$$

always represents a conic section, whose axes are parallel to the coördinate axes. There are three cases, corresponding to the three species of conic.

(1) The parabola: either A or B is zero. In exceptional cases this curve degenerates into a pair of real or imaginary parallel straight lines, and these may coincide. [Art. 107]

(2) The ellipse: neither A nor B is zero, and they have like signs. In exceptional cases this curve degenerates into a circle, a point, or an imaginary locus. [Art. 113, NOTE]

(3) The hyperbola: neither A nor B is zero, and they have unlike signs. In exceptional cases this curve degenerates into a pair of real intersecting lines. [Art. 119]

The ellipse and hyperbola have centers, and therefore are called **central conics**, while the parabola is said to be **non-central**; although it is at times more convenient to consider that the latter curve has a center at infinity, on the principal axis (cf. Appendix, Note E).

The equation for each conic has two standard forms, which state a characteristic geometric property of the curve, and to which all other equations representing that species can be reduced. These standard forms are the simplest for studying the curves; but the student must discriminate carefully between general results and those which hold only when the equation is in the standard form.

IV. TANGENTS, NORMALS, POLARS, DIAMETERS, ETC.

121. Since the equation

$$Ax^2 + By^2 + 2Gx + 2Fy + C = 0 \quad . \quad . \quad . \quad (1)$$

always represents a conic whose axes are parallel to the coördinate axes, and since by giving suitable values to the constants A, B, G, F, and C, equation (1) may represent *any* such conic, therefore, if the equations of tangents, normals, polars, etc., to the locus of equation (1) can be found, independent of the values that A, B, etc., may have, these equations will represent the tangents, etc., when any special values whatever are given to the constants involved.

In the next few articles such equations will be found.

122. Tangent to the conic

$$\boldsymbol{Ax^2 + By^2 + 2Gx + 2Fy + C = 0}$$

in terms of the coördinates of the point of contact: the secant method. The definition of a tangent has already been given (Art. 81), and the method to be employed here in finding its equation is the one which was used in Art. 84. That article should now be carefully re-read.

Let the given conic, *i.e.*, the locus of the equation,

$$Ax^2 + By^2 + 2Gx + 2Fy + C = 0, \quad . \quad . \quad . \quad (1)$$

be represented by the curve BHK; and let $P_1 \equiv (x_1, y_1)$ be the point of tangency.

Through $P_1 \equiv (x_1,\ y_1)$ draw a secant line LM, and let $P_2 \equiv (x_2, y_2)$ be its other point of intersection with the locus of equation (1). If the point P_2 moves along the curve until it comes into coincidence with P_1, the limiting position of the secant LM is the tangent P_1T.

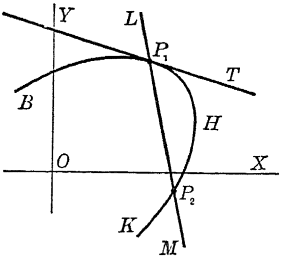


FIG. 90.

The equation of the line LM is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1). \quad . \quad . \quad . \quad . \qquad (2)$$

If now P_2 approaches P_1 until $x_2 = x_1$ and $y_2 = y_1$, equation (2) assumes the indeterminate form

$$y - y_1 = \frac{0}{0}(x - x_1). \quad . \quad . \quad . \quad . \qquad (3)$$

This indeterminateness arises because account has not yet been taken of the path (or direction) by which P_2 shall approach P_1, and it disappears immediately if the condition that P_1 and P_2 are points on the conic (1) is introduced. Since P_1 and P_2 are on the conic (1),

therefore $\quad Ax_1^2 + By_1^2 + 2\ Gx_1 + 2\ F_1y + C = 0, \ .\ .\ . \ (4)$

and $\quad Ax_2^2 + By_2^2 + 2\ Gx_2 + 2\ Fy_2 + C = 0, \ .\ .\ . \ (5)$

Subtracting equation (4) from equation (5), transposing, factoring, and rearranging [cf. Art. 84, equations (8), (9), and (10)], the result may be written

$$\frac{y_2 - y_1}{x_2 - x_1} = -\frac{A(x_1 + x_2) + 2\ G}{B(y_1 + y_2) + 2\ F}. \quad . \quad . \quad . \quad . \qquad (6)$$

If this value of $\frac{y_2 - y_1}{x_2 - x_1}$ is substituted in equation (2), the result is

$$y - y_1 = -\frac{A(x_1 + x_2) + 2\ G}{B(y_1 + y_2) + 2\ F}(x - x_1), \quad . \quad . \quad . \qquad (7)$$

which is the equation of the secant line LM of the given conic (1).

If now this secant line be revolved about P_1 until P_2 comes into coincidence with P_1, *i.e.*, until $x_2=x_1$ and $y_2=y_1$, this equation becomes

$$y - y_1 = -\frac{Ax_1 + G}{By_1 + F}(x - x_1); \quad . \quad . \quad . \quad (8)$$

which is, therefore, the equation of the tangent line P_1T at the point P_1. This equation (8) can be put in a much simpler and more easily remembered form, thus :

Clearing equation (8) of fractions, and simplifying, it may be written

$$Ax_1x + By_1y + Gx + Fy = Ax_1^2 + By_1^2 + Gx_1 + Fy_1, \quad . \quad . \quad . \quad (9)$$

but, from equation (3),

$$Ax_1^2 + By_1^2 + Gx_1 + Fy_1 = -Gx_1 - Fy_1 - C,$$

hence substituting this value in the second member of equation (9) that equation becomes

$$Ax_1x + By_1y + Gx + Fy = -Gx_1 - Fy_1 - C, \quad . \quad . \quad . \quad (10)$$

and, by transposing and combining, this may be written,

$$\boldsymbol{Ax_1x + By_1y + G(x+x_1) + F(y+y_1) + C = 0.}\text{*} \quad . \quad . \quad . \quad [50]$$

This is, then, the equation of the tangent to the conic

$$Ax^2 + By^2 + 2\,Gx + 2\,Fy + C = 0,$$

whatever the values of the coefficients A, B, G, F, and C may be ; the point (x_1, y_1) being the point of contact.

If $A=0$, $B=1$, $G=-2p$, $F=0$ and $C=0$, then the equation of this conic becomes $y^2=4px$, and the equation of the tangent becomes, $yy_1=2p(x+x_1)$; similarly for any other special form of the equation of the conic.

---

* Compare note, Art. 84, (β).

123. Normal to the conic $Ax^2 + By^2 + 2Gx + 2Fy + C = 0$, at a given point. The normal to a curve has been defined (Art. 81) as a straight line perpendicular to a tangent, and passing through the point of contact. Therefore, to obtain the equation of a normal to a conic, at a given point on the conic, it is only necessary to write the equation of the tangent to the conic at that point (by Art. 122), and then find the equation of a perpendicular to the tangent which passes through the point of contact (cf. Arts. 53, 62).

Example. To find the equation of the normal to the ellipse

$$\frac{x^2}{18} + \frac{y^2}{8} = 1$$

at the point (3, 2).

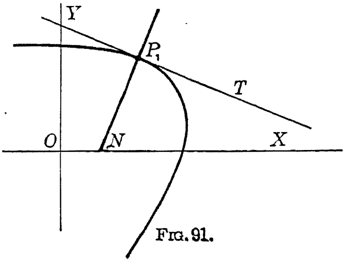


Fig. 91.

The equation of the tangent at the point (3, 2) is

$$\frac{3x}{18} + \frac{2y}{8} = 1;$$

i.e.,

$$2x + 3y = 12.$$

The perpendicular line through (3, 2) is

$$3x - 2y = 5,$$

which is, therefore, the required normal.

Similarly, to find the normal to the conic whose equation is

$$Ax^2 + By^2 + 2Gx + 2Fy + C = 0, \quad . \quad . \quad . \quad (1)$$

at the point $P_1 \equiv (x_1, y_1)$ on the curve. The equation of the tangent at P_1 is (Art. 122)

$$Ax_1x + By_1y + G(x + x_1) + F(y + y_1) + C = 0 \quad . \quad . \quad . \quad (2)$$

and its slope is, therefore, (Art. 58 (2))

$$-\frac{Ax_1 + G}{By_1 + F}.$$

Hence the required equation of the corresponding normal at P_1 is (Arts. 53, 62)

$$y - y_1 = \frac{By_1 + F}{Ax_1 + G}(x - x_1).^* \quad . \quad . \quad . \quad [51]$$

EXERCISES

1. Is the line $3x + 2y = 17$ tangent to the ellipse $16x^2 + 25y^2 = 400$?

2. Find the equation of a tangent to the conic $x^2 + 5y^2 - 3x + 10y - 4 = 0$, parallel to the line $y = 3x + 7$ (cf. Art. 82).

Write the equations of the tangent and normal to each of the following conics, through a point (x_1, y_1) on the curve (cf. Art. 122 [50]).

3. $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

4. $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

5. $x^2 = 4p(y - 5)$; sketch the figure.

6. $3x^2 - 5y^2 + 24x = 0$; sketch the figure.

7. $x^2 + 5y^2 - 3x + 10y - 4 = 0$; sketch the figure.

8. Derive, by the secant method (cf. Art. 122), the tangent to the parabola $y^2 = 4px$; the point of contact being (x_1, y_1).

9. Derive, by the secant method, the tangent to the ellipse $x^2 + 4y^2 - 8x + 20y = 0$; the point of contact being (x_1, y_1).

Write the equations of the tangents and normals to each of the following conics, at the given point; also sketch each figure:

10. $9x^2 + 5y^2 + 36x + 20y + 11 = 0$, at the point $(^-2, 1)$;

11. $9x^2 + 4y^2 + 6x + 4y = 0$, at the point $(0, 0)$;

12. $y^2 - 6y - 8x = 31$, at the point $(^-3, -1)$;

---

[*] Since the equation of the normal [51] is so readily deduced, in every particular case, from that of the tangent, and since the latter is so easily remembered, it is not recommended that equation [51] be memorized.

13. $\frac{x^2}{4}+\frac{y^2}{4}=1$, at the point $(1, \sqrt{3})$;

14. $3x^2+4y^2=16$, at the point $(2, {}^-1)$.

124. **Equation of a tangent, and of a normal, that pass through a given point which is not on the conic.**

The method to be followed in finding the equation of a tangent, or of a normal, that passes through a given point which is *not on the conic*, may be illustrated by the following example; the same method is applicable to any conic whatever.

Let it be required to find the equation of that tangent to the parabola

$$y^2-6y-8x-31=0, \quad \cdot \quad \cdot \quad \cdot \qquad (1)$$

which passes through the point $({}^-4, {}^-1)$. This point not being *on* the parabola, the method of Art. 116 does not apply; but, assuming for the moment that it is possible to draw such a tangent, let (x_1, y_1) be its point of contact. The equation of this tangent is (Art. 122)

$$y_1y-3(y+y_1)-4(x+x_1)-31=0. \quad \cdot \quad \cdot \quad \cdot \qquad (2)$$

Since this tangent passes through the point $({}^-4, {}^-1)$, therefore equation (2) is satisfied by the coördinates $^-4$ and $^-1$,

i.e., $$-y_1-3(-1+y_1)-4(-4+x_1)-31=0, \quad \cdot \quad \cdot \quad \cdot \qquad (3)$$

which reduces to $$x_1+y_1+3=0. \quad \cdot \quad \cdot \quad \cdot \qquad (4)$$

Equation (4) furnishes *one* relation between the two unknown constants x_1 and y_1; another equation between these two unknowns is furnished by the fact that (x_1, y_1) is a point on the parabola (1); this equation is

$$y_1^2-6y_1-8x_1-31=0. \quad \cdot \quad \cdot \quad \cdot \qquad (5)$$

Solving between equations (4) and (5) gives

$$x_1=-2\pm 2\sqrt{2} \text{ and } y_1=-1\mp 2\sqrt{2};$$

hence, there are *two* points on the given parabola the tangents at which pass through the point $({}^-4, {}^-1)$; their coördinates are $(-2+2\sqrt{2}, -1-2\sqrt{2})$ and $(-2-2\sqrt{2}, -1+2\sqrt{2})$; and substituting either pair of these values for x_1 and y_1 in equation (2) gives the equation of a straight line that is tangent to the parabola (1), and that passes through the point $({}^-4, {}^-1)$.

So, too, if it is desired to find the equation of a normal through a point *not on* the curve, it is only necessary to *assume* temporarily the coördinates of the point *on* the curve through which this normal passes, and then *find* these coördinates by solving two equations, corresponding to equations (4) and (5) above.

The problem of finding the above tangent could also have been solved by writing the equation of a line through the point $(^-4, ^-1)$ (Art. 53) and having the undetermined slope m, and then so determining m that the two points in which this line meets the parabola should be coincident.

125. Through a given external point two tangents to a conic can be drawn. This theorem can be proved in precisely the same way as the corresponding theorem in the case of the circle (Art. 89) was proved. It may also be proved by the method already applied to the parabola in the preceding article. Let the latter method be adopted. Suppose the equation of the conic to be

$$Ax^2 + By^2 + 2\,Gx + 2\,Fy + C = 0; \quad . \quad . \quad . \quad (1)$$

Let the locus of this equation be represented by the curve LP_1P_2L', and let $Q \equiv (h, k)$ be the given external point.

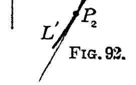


FIG. 92.

If $P_1 \equiv (x_1, y_1)$ is a point on LP_1P_2L', then the equation of the tangent at P_1 is

$$Ax_1x + By_1y + G(x + x_1) + F(y + y_1) + C = 0, \quad (2)$$

and this tangent will pass through the point Q if

$$Ahx_1 + Bky_1 + G(h + x_1) + F(k + y_1) + C = 0. \quad (3)$$

But P_1 being on the locus of equation (1), its coördinates x_1 and y_1 also satisfy equation (1);

i.e., $$Ax_1^2 + By_1^2 + 2\,Gx_1 + 2\,Fy_1 + C = 0. \quad . \quad . \quad . \quad (4)$$

If now equations (3) and (4) are solved for x_1 and y_1, *two* values of each are found; these values are both imaginary f Q is *within* the conic, they are real but coincident if Q is *on* the conic, and they are real and distinct if Q is *outside* of the conic. This proves not only the above proposition but also the fact that no real tangent can be drawn to a conic through an internal point, and that only one tangent can be drawn to a conic through a given point on the curve.

126. Equation of a chord of contact. If the two tangents are drawn from an *external* point to a conic section, the straight line through the corresponding points of tangency is called the **chord of contact** corresponding to the point from which the tangents are drawn (cf. Art. 90).

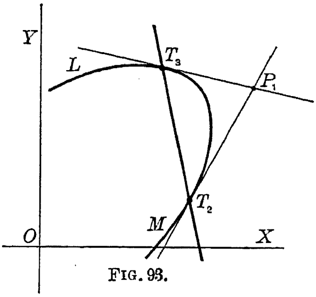


FIG. 93.

Let $P_1 \equiv (x_1, y_1)$ be the external point from which the two tangents are drawn; $T_2 \equiv (x_2, y_2)$ and $T_3 \equiv (x_3, y_3)$, the points of tangency of these tangents to the conic whose equation is

$$Ax^2 + By^2 + 2\,Gx + 2\,Fy + C = 0; \quad . \quad . \quad . \quad (1)$$

it is required to find the equation of the line through T_2 and T_3.

The equation of the tangent at T_2 (cf. Art. 122) is

$$Ax_2x + By_2y + G(x + x_2) + F(y + y_2) + C = 0, \; . \; . \; . \; (2)$$

and the equation of the tangent at T_3 is

$$Ax_3x + By_3y + G(x + x_3) + F(y + y_3) + C = 0. \; . \; . \; . \; (3)$$

Since each of these tangents, by hypothesis, passes through P_1, therefore the coördinates x_1 and y_1 satisfy both equation (2) and equation (3); *i.e.*,

$$Ax_1x_2 + By_1y_2 + G(x_1 + x_2) + F(y_1 + y_2) + C = 0, \ldots (4)$$

and $Ax_1x_3 + By_1y_3 + G(x_1 + x_3) + F(y_1 + y_3) + C = 0. \quad (5)$

Equations (4) and (5), respectively, assert that the points

$$T_2 \equiv (x_2, y_2) \quad \text{and} \quad T_3 \equiv (x_3, y_3)$$

are each on the locus of the equation

$$\boldsymbol{A x_1 x + B y_1 y + G(x + x_1) + F(y + y_1) + C = 0}. \quad . \quad . \quad [52]$$

But equation [52] is of the first degree in the two variables x and y, hence (Art. 57) its locus is a straight line; *i.e.*, [52] is the equation of the straight line through T_2 and T_3, which was to be found.

NOTE 1. The equation [52] of the chord of contact corresponding to a given external point (x_1, y_1), and the equation [50] of the tangent whose point of contact is (x_1, y_1) are identical in form. This might have been expected because the tangent is only a special case of the chord of contact, since the chord of contact, for a given point, approaches more and more nearly to coincidence with a tangent when the point is taken more and more nearly on the curve.

NOTE 2. The present article furnishes another method of treatment for the question of Art. 124. To get the equations of the two tangents that can be drawn through a given external point to a given conic, it is only necessary to write the equation of the chord of contact corresponding to this point; then find the points in which this chord of contact intersects the conic. These are the points of contact of the required tangents, whose equation may then be written down.

EXERCISES

1. By first finding the chord of contact (Art. 126) of the tangents drawn from the point $(-\frac{4}{3}, \frac{13}{3})$ to the conic

$$4x^2 + y^2 + 24x - 2y + 17 = 0,$$

find the points of contact, and then write the equations of the tangents to the conic at these points; verify that these two tangents intersect in the point $(-\frac{4}{3}, \frac{13}{3})$.

2. Solve Ex. 1 by the method of Art. 124.

3. Solve Ex. 1 by the method of Art. 89.

4. Find the equation of a normal through the point (7, 5) to the conic

$$4x^2 + y^2 + 24x - 2y + 17 = 0.$$

Is it possible to draw more than one normal through (7, 5) to the given conic?

5. By the methods of Exs. 1, 2, and 3, find the equations of the tangents through the origin to the conic

$$3x^2 - 2y^2 = 6x + 8y + 6.$$

6. By the methods of Exs. 1, 2, and 3, find the equations of the tangents through the point (−1, 1) to the conic

$$9x^2 + 5y^2 + 30x + 20y + 11 = 0.$$

7. Sketch the conics whose equations are given in Ex. 1, 5, and 6.

8. Find the equations of the tangents to the conic, $x^2 + 4y^2 = 4$, from the point (3, 2).

9. Find the normal to the conic $x^2 + 4y^2 = 4$, through the point (3, 2).

10. Solve Exs. 8 and 9, by assuming the slope m of the required line (Art. 53), and then determining m so that the two points in which the line meets the given curve shall be coincident.

127. Poles and polars. If through any given point $P_1 \equiv (x_1, y_1)$, outside, inside, or on a given conic, a secant is drawn, meeting the conic in two points Q and R, and if tangents at Q and R are drawn, they will intersect in some point, as $P' \equiv (x', y')$. The locus of P' as the secant revolves about P_1 is the **polar** of the point P_1 (cf. Art. 91) with regard to the given conic; and P_1 is the **pole** of that locus.

To find the equation of the polar of a given point

$$P_1 \equiv (x_1, y_1),$$

with regard to a given conic whose equation is

$$Ax^2 + By^2 + 2Gx + 2Fy + C = 0, \quad . \quad . \quad . \quad (1)$$

let QP_1R be any position of the secant through P_1, and

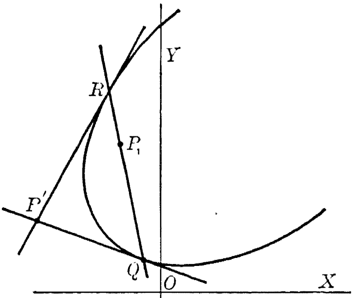


FIG. 94.

let the tangents at Q and R intersect in $P' \equiv (x', y')$. Then the equation of QP_1R (Art. 126) is

$$Ax'x + By'y + G(x + x') + F(y + y') + C = 0. \dots (2)$$

Since this line passes through P_1, therefore the coördinates x_1 and y_1 satisfy equation (2),

i.e., $Ax_1x' + By_1y' + G(x_1 + x') + F(y_1 + y') + C = 0, \dots (3)$

and equation (3) asserts that the variable point $P' \equiv (x', y')$ lies on the locus of the equation

$$Ax_1x + By_1y + G(x + x_1) + F(y + y_1) + C = 0. \dots (4)$$

Equation (4) is of the first degree in the variables x and y, hence (Art. 57), its locus is a straight line; the polar of P_1, with regard to the conic (1), *i.e.*, the locus of P', is then the straight line whose equation is

$$\boldsymbol{Ax_1x + By_1y + G(x + x_1) + F(y + y_1) + C = 0.} \dots [53]$$

NOTE. That the equation of a tangent [50] and of a chord of contact [52] have the same form as equation [53] is due to the fact that a tangent, and a chord of contact, are but special cases of a polar.

128. Fundamental theorem. An important theorem concerning poles and polars is: *If the polar of the point P_1, with*

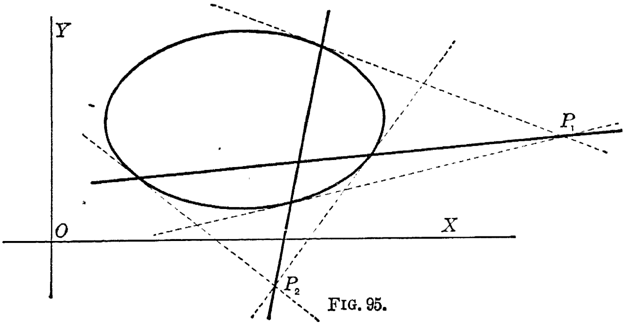


FIG. 95.

regard to a given conic, passes through the point P_2, then the polar of P_2 with regard to the same conic passes through P_1.

Let the equation of the given conic be

$$Ax^2 + By^2 + 2\,Gx + 2\,Fy + C = 0, \quad . \quad . \quad . \quad (1)$$

and let the two given points be

$$P_1 \equiv (x_1, y_1) \text{ and } P_2 \equiv (x_2, y_2).$$

Then the equation of the polar of P_1 with regard to the conic (1) is (Art. 127)

$$Ax_1x + By_1y + G(x + x_1) + F(y + y_1) + C = 0; \quad . \quad . \quad . \quad (2)$$

if this line passes through P_2, then

$$Ax_2x_1 + By_2y_1 + G\,(x_2 + x_1) + F(y_2 + y_1) + C = 0. \quad . \quad . \quad (3)$$

But the polar of P_2 with regard to the conic (1) is

$$Ax_1x + By_1y + G(x + x_2) + F(y + y_2) + C = 0, \quad . \quad . \quad . \quad (4)$$

and equation (3) shows that the locus of equation (4) passes through the point P_1, which proves the proposition.

129. Diameter of a conic section. The locus of the middle points of any system of parallel chords of a given conic is called a **diameter** of that conic, and the chords which that diameter bisects are called the **chords of that diameter.**

For a given conic, it is required to find the equation of the diameter bisecting a system of chords whose slope is m. Let the equation of the given conic (HJK, Fig. 96) be

$$Ax^2 + By^2 + 2\,Gx + 2\,Fy + C = 0, \quad . \quad . \quad . \quad (1)$$

let the equation of any one of the parallel chords of slope m, LM for example, be

$$y = mx + b, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and let the two points in which it meets the given conic be

$$P_1 \equiv (x_1, y_1) \text{ and } P_2 \equiv (x_2, y_2).$$

Then (Art. 122, eq. (6)),

$$= -\frac{A(x_1 + x_2) + 2G}{B(y_1 + y_2) + 2F}. \quad . \quad . \quad . \quad (3)$$

If $Q \equiv (h, k)$ be the middle point of the chord P_1P_2, then

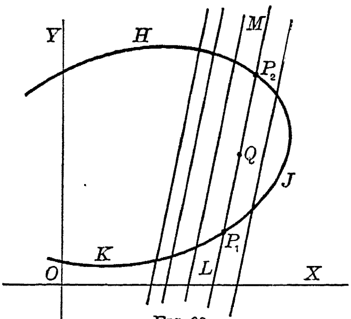


FIG. 96.

$$= \frac{x_1 + x_2}{2} \text{ and } k = \frac{y_1 + y_2}{2};$$

substituting these values of $x_1 + x_2$ and $y_1 + y_2$ in equation (3), then clearing of fractions and transposing, that equation becomes

$$Ah + mBk + G + mF = 0. \quad . \quad . \quad . \quad (4)$$

But equation (4) asserts that the coördinates (h, k) of the middle point of any one of this system of parallel chords satisfy the equation

$$Ax + mBy + G + mF = 0, \quad . \quad . \quad . \quad [54]$$

which is therefore the equation of the diameter whose chords have the slope m.

EXERCISES

1. Find the polar of the point (2, 1) with regard to the hyperbola $x^2 - 2(y^2 + x) - 4 = 0$. Show that this polar passes through (12, 3), and then verify Art. 128, for this particular case, by showing that the polar of (12, 3), with regard to the given hyperbola, passes through (2, 1).

2. Write the equation of the chord of contact of the tangents drawn through (2, 1) to the hyperbola $x^2 - 2y^2 - 2x - 4 = 0$, then find the points in which it meets the curve, get the equations of the tangents at these points, and verify that they pass through the given point (2, 1).

3. By specializing the coefficients in equation [54], prove that the diameter of a circle is perpendicular to the chords of that diameter.

SOLUTION. If equation (1) of Art. 123 represents a circle, then $A = B$, and then equation [54] becomes

$$y = -\frac{1}{m}x - \frac{G + mF}{Am},$$

i.e., the slope of the diameter is $-\frac{1}{m}$; but the slope of the given system of chords is m, hence the diameter is perpendicular to its chords.

4. By means of eq. [54], *i.e.*, by specializing its coefficients, prove that the diameter of a circle passes through the center of the circle.

5. By means of equation [54] prove that any diameter of the ellipse $3x^2 + y^2 - 6x + 2y = 0$ passes through the center of the ellipse. Does this property belong to all ellipses? To all conics?

6. Find the equation of that diameter of the hyperbola

$$x^2 - 4y^2 + 16y + 6x - 15 = 0,$$

whose chords are parallel to the line $y = 2x + 10$. Does this diameter pass through the center of the curve?

7. Find the angle between the diameter and its chords in exercise 6.

8. Show that every diameter of the parabola $3y^2 - 16x + 12y = 4$ is parallel to its axis. Is this a property belonging to all parabolas?

9. Derive, by the method of Art. 129, the equation of that diameter of the hyperbola $x^2 - 4y^2 + 16y + 6x - 15 = 0$, which bisects chords parallel to the line $3x - 4y = 12$.

130. Equation of a conic that passes through the intersections of two given conics. Let the given conics be

$$S_1 \equiv A_1x^2 + B_1y^2 + 2G_1x + 2F_1y + C_1 = 0, \quad . \quad . \quad . \quad (1)$$

and $$S_2 \equiv A_2x^2 + B_2y^2 + 2G_2x + 2F_2y + C_2 = 0; \quad . \quad . \quad . \quad (2)$$

then, if k be any constant whatever,

$$S_1 + kS_2 = 0 \quad . \quad . \quad . \quad (3)$$

represents a conic whose axes are parallel to the coördinate axes (Art. 120), and which passes through the points in which the conics $S_1 = 0$ and $S_2 = 0$ intersect each other (Art. 41); *i.e.*, $S_1 + kS_2 = 0$ represents a *family* of conics, each member of which passes through the intersections of $S_1 = 0$ and $S_2 = 0$. The parameter k may be so chosen that the conic (3) shall, in addition to passing through the four points in which $S_1 = 0$ and $S_2 = 0$ intersect, satisfy one other condition; *e.g.*, that it shall pass through a given fifth point.

Moreover, if $S_1 = 0$ and $S_2 = 0$ are both circles, then $S_1 + kS_2 = 0$ is also a circle (cf. Arts. 95 and 96).

V. POLAR EQUATION OF THE CONIC SECTIONS

131. Polar equation of the conic. Based upon the "focus and directrix" definition already given in Art. 48, the polar equation of a conic section is easily derived.

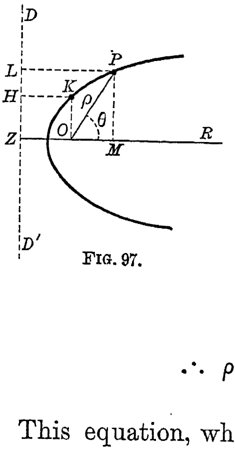


FIG. 97.

Let $D'D$ (Fig. 97) be the given line (the directrix) and O the given point (the focus); draw ZOR through O and perpendicular to $D'D$, and let O be chosen as the pole and OR as the initial line. Also let $P \equiv (\rho, \theta)$ be any point on the locus, and let e be the eccentricity. Draw MP and OK parallel, and LP and HK perpendicular, to $D'D$, and let $OK = l$; then

$$OP = e \cdot LP, \quad \text{[definition of the curve]}$$

$$= e(ZO + OM);$$

$$\therefore \quad \rho = e\left(\frac{l}{e} + \rho\cos\theta\right).$$

This equation, when solved for ρ, may be written in the form

$$\rho = \frac{l}{1 - e\cos\theta}, \quad \cdot \quad \cdot \quad \cdot \qquad [55]$$

which is the polar equation of a conic section referred to its focus and principal axis; e being the eccentricity and l the semi-latus-rectum. If $e = 1$, equation [55] represents a parabola; if $e < 1$, an ellipse; and if $e > 1$, an hyperbola.

NOTE. Equation [55] shows that if $e < 1$, *i.e.*, if the equation represents an ellipse, there is no value of θ for which ρ becomes infinite. Therefore there is *no* direction in which a line may be drawn to meet an ellipse at infinity. If $e = 1$, *i.e.*, if the equation represents a parabola, there is one value of θ, viz., $\theta = 0$, for which ρ becomes infinite. Therefore there is *one* direction in which a line may be drawn to meet a parabola at infinity. If $e > 1$, *i.e.*, if the equation represents an hyperbola, there are two values of θ, viz., $\theta = \pm \cos^{-1}(1 : e)$, for which ρ becomes infinite. Therefore there are *two* directions in which a line may be drawn to meet an hyperbola at infinity.

The three species of conic sections may therefore be distinguished from each other by the number of directions in which lines may be drawn through the focus to meet the curve at infinity. Or, since parallel lines meet at infinity, any point of the plane may be used instead of the focus.

132. From the polar equation of a conic to trace the curve. Suppose $e > 1$, *i.e.*, suppose equation [55] represents an hyperbola. When $\theta = 0$, $\rho = \frac{l}{1-e}$, hence ρ is negative; as θ increases, $\cos\theta$ decreases, and $e\cos\theta$ becomes numerically more and more nearly equal to 1; therefore ρ remains negative and becomes larger and larger; $\rho = -\infty$ when

$$1 - e\cos\theta = 0,$$

i.e., when

$$\theta = \cos^{-1}\left(\frac{1}{e}\right) = \alpha,$$

say; as θ increases through this value, ρ becomes $+\infty$ and then decreases, but remains positive, and becomes equal to l when $\theta = 90°$; as θ increases through 90° to 180°, ρ remains positive, but continues to decrease, reaching its smallest value, viz. $\rho = \frac{l}{1+e}$, when $\theta = 180°$; as θ increases from 180° to 270°, ρ remains positive and increases from $\frac{l}{1+e}$ to l; as θ increases from 270° to $360° - \alpha$, ρ increases from l to $+\infty$; as θ increases through $360° - \alpha$, ρ becomes $-\infty$; and finally, as θ increases from $360° - \alpha$ to 360°, ρ remains negative, but decreases numerically, reaching the value $\frac{l}{1-e}$ again when θ becomes 360°.

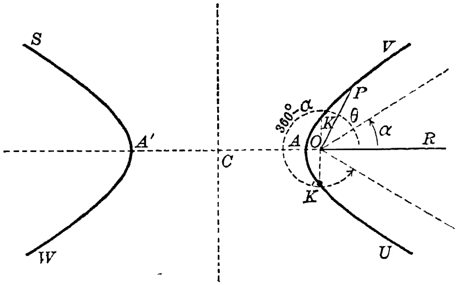


FIG. 98.

These deductions from equation [55] show that the hyperbola has the form represented in Fig. 98, and that, as θ increases from 0 to α, the lower half $A'W$ of the infinite branch at the left is traced; as θ increases from α to $360° - \alpha$, the right hand branch VAU is traced; and as θ increases from $360° - \alpha$ to 360°, the upper half SA' of the left hand branch is traced.

If θ increases beyond 360°, the tracing point moves along the same curve; this is also true if θ changes from 0° to $-360°$.

NOTE. To show the identity of the curve as traced in the present article and in Art. 117, it need only be recalled that

$$e = \frac{\sqrt{a^2+b^2}}{a}, \text{ and that } l = \frac{b^2}{a}.$$

These values substituted above show that

$$\alpha = \cos^{-1}\left(\frac{a}{\sqrt{a^2+b^2}}\right) = \tan^{-1}\left(\frac{b}{a}\right), \text{ that } OA' = -\left(a + \sqrt{a^2+b^2}\right), \text{ etc.}$$

EXERCISES

1. From equation [55], trace the parabola.

2. From equation [55], trace the ellipse.

3. By means of equation [55], prove that the length of a chord through the focus of a parabola, and making an angle of 30° with the axis of the curve, is four times the length of the latus-rectum.

4. By transforming from rectangular to polar coördinates, derive the polar equations of the conic sections from their rectangular equations.

EXAMPLES ON CHAPTER VIII

1. Find the equations of those tangents to the conic $7x^2 - 12y^2 = 112$, which pass through the point $(-9, 7)$.

2. What is the polar of the point $(7, 2)$ with reference to the conic $16y^2 + 9x^2 = 144$? Find the equation of the line which is tangent to the conic and parallel to this polar.

3. Find the polars of the foci of the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$, with regard to this ellipse. Also for the parabola $y^2 = 4px$.

4. What is the equation of the polar of the center of the conic $Ax^2 + By^2 + 2Gx + 2Fy + C = 0$, with reference to the conic?

5. What is the pole of the directrix of the hyperbola $x^2 - 4y^2 = 16$, with reference to that curve?

6. The line $y = m(x - ae)$ passes through the focus of the central conic $\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$. On what line does its pole lie? Find the line joining its pole to the focus. What relation exists between this line and the given focal chord?

7. What is the polar of the vertex of the conic

$$Ax^2 + By^2 + 2\,Gx + 2\,Fy + C = 0,$$

with reference to the curve?

8. What is the equation of each common chord of the two conics

$$16\,x^2 + 9\,y^2 = 144, \qquad 16\,x^2 - 9\,y^2 = 144?$$

HINT. Use Art. 130, equation 3; find k so that $S_1 + kS_2$ can be factored.

9. Prove that the perpendicular dropped from any point of the directrix, to the polar of that point, passes through the focus

$$(\alpha)\ \text{for } y^2 = 4\,px. \qquad (\beta)\ \text{for } \frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1.$$

Using the simplest standard equations of the conics, find for each

10. the polar of the focus;

11. the pole of the directrix;

12. the ratio of the angle subtended by a chord at its pole, and the angle subtended by the same chord at the focus.

13. Find a conic through the intersections of the ellipse $4\,x^2 + y^2 = 16$ and the parabola $y^2 = 4\,x + 4$, and also passing through the point 2, 2. What kind of a conic is it?

14. Show that the curves $\frac{x^2}{16} + \frac{y^2}{7} = 1$ and $\frac{x^2}{4} - \frac{y^2}{5} = 1$ have the same foci, and that they cut each other at right angles.

15. Find the vertices of an equilateral triangle circumscribed about the ellipse $9\,x^2 + 16\,y^2 = 144$, one side being parallel to the major axis of the curve.

16. Find the normal to the conic $3\,x^2 + y^2 - 2\,x - y = 1$, making the angle $\tan^{-1}(\frac{3}{4})$ with the x-axis.

17. Show that the locus of the pole, with respect to the parabola $y^2 = 4\,ax$, of a tangent to the hyperbola $x^2 - y^2 = a^2$, is the ellipse $4\,x^2 + y^2 = 4\,a^2$.

18. Show that $\frac{x^2}{a^2 - k^2} + \frac{y^2}{b^2 - k^2} = 1$, where k is an arbitrary constant, represents an ellipse having the same foci as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ when

$k^2 < b^2$; but represents a confocal hyperbola when $a^2 > k^2 > b^2$; given $a > b$.

Determine the nature of the following conics; and also their foci, directrices, centers, semi-axes, and latera recta:

19. $y^2 = (x + 3)(x + 4)$;

20. $x^2 - 4y^2 + x + y + 1 = 0$;

21. $x^2 = 4x + 11y + 7$;

22. $3x^2 + y^2 - 6x + 8y + 1 = 0$;

23. $3x^2 + 5y = 3y^2 + 5x$;

24. $9(x^2 - y) = 3y(1 + 2x - 3y)$.

25. Show that the polar equation of the parabola, with its vertex at the pole, is $\rho = \frac{4p \cos\theta}{\sin^2\theta}$.

26. Show that if the left hand focus be taken as pole, the polar equation of the ellipse is $\rho = \frac{a(1 - e^2)}{1 - e\cos\theta}$.

27. Derive the polar equation of an hyperbola, with its pole at the focus, eccentricity 2, and the distance of the focus from the directrix equal to 6.

CHAPTER IX

THE PARABOLA $y^2 = 4px$

133. Review. In the preceding chapter (Arts. 102 to 108), the nature of the parabola has been examined, and its equation derived in two standard forms. These equations are:

$y^2 = 4px$, if the axis of the curve coincides with the x-axis, and the tangent at the vertex with the y-axis; and

$(y - k)^2 = 4p(x - h)$, if the axis of the curve is parallel to the x-axis, and the vertex is at the point (h, k). In the present chapter, some of the intrinsic properties of the parabola are to be studied, *i.e.*, properties which belong to the curve and are entirely independent of the position of the coördinate axes. For this purpose, it will, in general, be easier to use the simplest form of the equation of the curve, viz., $y^2 = 4px$.

In every parabola, the value of the eccentricity is $e = 1$. If the equation of the parabola is $y^2 = 4px$, then the focus is the point $(p, 0)$, the directrix is the line $x = -p$, and the axis of the curve is the line $y = 0$. The equation

$$y_1 y = 2p(x + x_1)$$

represents the polar of the point $P_1 \equiv (x_1, y_1)$ with respect to the parabola, for all positions of P_1. If P_1 be outside the curve, this polar is the chord of contact corresponding to tangents from P_1; if P_1 be upon the curve, this polar is the tangent at that point. These facts, shown in the previous chapter, will be assumed in the following discussion.

134. Construction of the parabola. The two conceptions of a locus given in Article 35 lead to two methods for constructing a curve, viz., by plotting points to be connected by a smooth curve, and by the motion of a point constrained by some mechanical device to satisfy the law which defines the curve. These two methods may be used in constructing a parabola.

(α) *By separate points.* Given the focus F and the vertex O, draw the axis OFX, the directrix $D'D$ cutting this axis in Z, and also a series of lines perpendicular to the axis at M_1, M_2, M_3, etc., respectively. With F as center and ZM_1 as radius, describe arcs cutting the line at M_1 in two points P_1 and Q_1; similarly, with F as center and ZM_2 as radius, cut the line at M_2 in P_2 and Q_2; and so on. The points thus found evidently satisfy the definition of the parabola (Art. 102). In this way, as many points of the curve as are desired may be found. If these be then connected by a smooth curve, it will be approximately the required parabola (cf. Note B, Appendix).

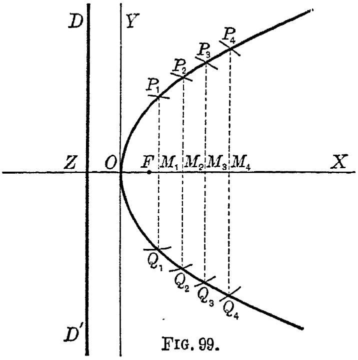


Fig. 99.

(β) *By a continuously moving point.* Let $D'D$ be the directrix and F the focus. Place a right triangle with its longer side KH in coincidence with the axis of the curve, and its shorter side KJ in coincidence with the directrix. Let one end of a string of length KH be fastened at

H, and the other end at *F*. If now a pencil point be pressed against the string, keeping it taut while the triangle is moved along the directrix, as indicated in the figure, then, in every position of *P*,

$$FP = KP,$$

therefore the pencil will trace an arc of a parabola.

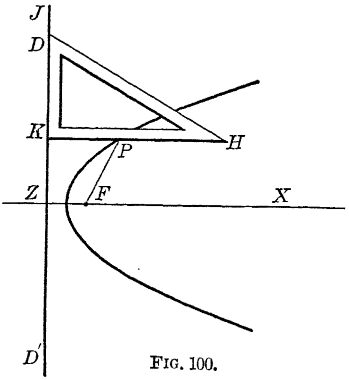


FIG. 100.

135. The equation of the tangent to the parabola $y^2 = 4px$ in terms of its slope. The equation of a line having the given slope m is

$$y = mx + k; \quad . \quad . \quad . \qquad (1)$$

it is desired to find that value of k for which this line will become tangent to the parabola whose equation is

$$y^2 = 4px. \quad . \quad . \quad . \qquad (2)$$

Considering equations (1) and (2) as simultaneous, and eliminating y, the resulting equation, which is

$$(mx + k)^2 = 4px, \quad . \quad . \quad . \qquad (3)$$

has for its roots the abscissas of the two points in which the loci of equations (1) and (2) intersect. These roots will become equal (cf. Art. 9), and therefore the points of intersection will become coincident, if

$$(mk - 2p)^2 - m^2k^2 \quad 0,$$

i.e., if

$$k = \frac{p}{m}. \quad . \quad . \quad . \qquad (4)$$

Therefore

$$y = mx + \frac{p}{m} \quad . \quad . \quad . \qquad [56]$$

is, for all values of m, the equation of a tangent to the parabola

$$y^2 = 4px.$$

The abscissa of the point of contact of the loci of equations (2) and [56] may be found from equation (3), by substituting in it the value of k given in equation (4); it is $\frac{p}{m^2}$. The ordinate may then be found from equation (1); it is $\frac{2p}{m}$. The point of contact is then $\left(\frac{p}{m^2}, \frac{2p}{m}\right)$.

136. The equation of the normal to the parabola $y^2 = 4px$ in terms of its slope. Since, by definition, the normal to a curve is perpendicular to the tangent at the point of contact, the equation of a normal to the parabola

$$y^2 = 4px \quad . \quad . \quad . \quad (1)$$

is, if m' be the slope of the tangent [Arts. 62, 135],

$$\left(y - \frac{2p}{m'}\right) = -\frac{1}{m'}\left(x - \frac{p}{m'^2}\right). \quad . \quad . \quad (2)$$

If m be the slope of the normal, then

$$m = -\frac{1}{m'},$$

and equation (2) may be written

$$y = mx - 2pm - pm^3. \quad . \quad . \quad . \quad [57]$$

This is the equation of a normal in terms of its own slope m.

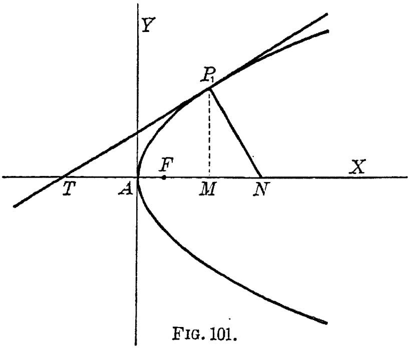


Fig. 101.

137. Subtangent and subnormal. Construction of tangent and normal. Let $P_1 \equiv (x_1, y_1)$ be any given point on the parabola whose equation is

$$y^2 = 4px. \quad . \quad . \quad . \quad (1)$$

Draw the ordinate MP_1, the tangent TP_1, and the normal P_1N.

Then by the definitions of Art. 86, the subtangent is TM, the subnormal is MN, the tangent length TP_1, and the normal length P_1N. The tangent at P_1 has for its equation (Art. 122),

$$y_1y = 2p(x + x_1), \quad . \quad . \quad . \quad (2)$$

hence its x-intercept is $AT = -x_1$. But $AM = x_1$,

therefore $$TM = 2x_1.$$

This proves that *the subtangent of the parabola* $y^2 = 4px$ *is bisected at the vertex;* and that *its length is equal to twice the abscissa of the point of contact.*

The normal at P_1 has for its equation (Art. 123)

$$y - y_1 = -\frac{y_1}{2p}(x - x_1), \quad . \quad . \quad . \quad (3)$$

hence its x-intercept is $AN = x_1 + 2p$. But $AM = x_1$,

therefore $$MN = 2p.$$

That is, in words, *the subnormal of the parabola* $y^2 = 4px$ *is constant; it is equal to half the latus rectum.*

These properties of the subtangent and subnormal give two simple methods of constructing the tangent and normal to any parabola at a given point, if the axis of the parabola is given.

First method: from the given point, let fall a perpendicular P_1M to the axis of the parabola, meeting it in M. The vertex of the curve being at A, construct the point T on the axis produced, so that $TA = AM$. The straight line TP_1 is the required tangent at P_1, and a line through P_1 at right angles to this tangent is the required normal.

Second method: from the foot of the perpendicular MP_1 construct the point N, so that MN equals twice the distance from vertex to the focus ($2p = 2AF$); then P_1N is the required normal, and a line through P_1 at right angles to P_1N is the required tangent.

EXERCISES

1. Construct a parabola with focus 2^{cm} from the directrix.

2. Construct a parabola with latus rectum equal to 6.

3. Find the equations of the two tangents to the parabola $y^2 = 4px$, which form with the tangent at the vertex a circumscribed equilateral triangle. Find also the ratio of the area of this triangle to the area of the triangle whose vertices are the points of tangency.

4. Find the equation of a tangent to the parabola $y^2 = 4px$, perpendicular to the line $4y - x + 3 = 0$, and find its point of contact.

5. Find the equations of the two tangents to the parabola $y^2 = 5x$ from the point (7, 1), using formula [56].

6. Write the equations of the tangents to the parabola $y^2 = 10x$, at the extremities of the latus rectum. On what line do these tangents intersect? (cf. Art. 138 (5), p. 228.)

7. Write the equations of the tangent and normal to the parabola $y^2 = 9x$, at the point (4, 6).

8. Write the equation of the normal to the parabola $y^2 = 6x$, drawn through the point $(\frac{3}{2}, 3)$.

9. Write the equation of the tangent to the parabola $y^2 = 4px$, for the point for which the normal length is twice the subtangent; for the point for which the normal length is equal to the difference between the subtangent and subnormal.

10. Two equal parabolas have the same vertex, and their axes are at right angles; find the equation of their common tangent, and show that the points of contact are each at the extremity of a latus rectum.

11. Find the locus of the middle point of the normal length of the parabola $y^2 = 4px$.

12. The subtangent of a parabola for the point (5, 4) is 10; find the equation of the curve, and length of the subnormal.

13. Find the subtangent, and the normal length, for the point whose abscissa $= -6$, and which is on the parabola $y^2 = -6x$.

14. Find the equation of the tangent parallel to the polar of $(^-1, 2)$ with respect to the parabola $y^2 = 12x$; also find the point of contact, the length of the tangent, and the subtangent.

15. Find the equation of a parabola which is tangent to the line $2y - 3x = 1$, and whose axis is parallel to the x-axis.

16. Show that the sum of the subtangent and subnormal for any point on the parabola $y^2=4px$, equals one half the length of focal chord parallel to the corresponding tangent.

17. Show that as the abscissa in the parabola $y^2 = 4px$ increases from 0 to ∞, the slope of the tangent diminishes from ∞ to 0; hence the curve is concave toward its axis.

138. Some properties of the parabola which involve tangents and normals. Let F be the focus, A the vertex, AX the axis, and $D'D$ the directrix of the parabola whose equation is

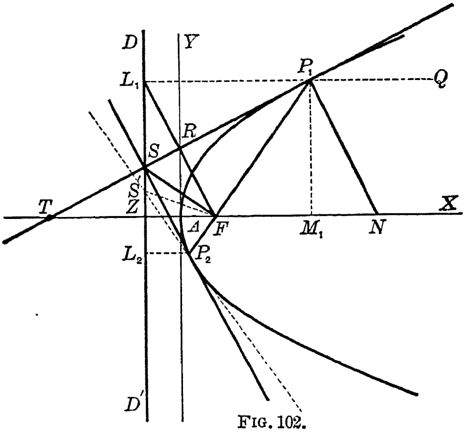


FIG. 102.

$$y^2 = 4px. \quad . \quad . \quad . \quad (1)$$

Through any point $P_1 \equiv (x_1, y_1)$ on the curve draw the tangent TP_1, cutting the y-axis in R, the directrix in S, and the x-axis in T; also draw the normal P_1N; the focal chord P_2FP_1; the tangent at P_2; the lines L_1P_1Q and L_2P_2, perpendicular to the directrix; and the lines SF and L_1F. Then the following properties of the parabola are readily obtained:

(1) *The focus is equidistant from the points* P_1, T, *and* N.

For $FP_1 = L_1P_1 = ZA + AM_1 = x_1 + p$,

$TF = TA + AF = x_1 + p$, Art. 137

and $FN = AM_1 + (M_1N - AF) = x_1 + p$; Art. 137

hence $FP_1 = TF = FN$.

The point F is the midpoint of the hypotenuse of the right triangle TP_1N, and is therefore equidistant from the vertices T, P_1, and N. Thus a third method is suggested for constructing the tangent and normal at P_1, viz.: by means of a circle, with the focus F as center, and the focal radius FP_1 as radius, which cuts the axis in T and N.

(2) *The tangent and normal bisect internally and externally, respectively, the angle between the focal radius to the point of contact and the perpendicular from that point to the directrix.*

For $\angle L_1P_1T = \angle P_1TF$, since $L_1P_1 \parallel TF$;

and $\angle TP_1F = \angle P_1TF$, since $TF = FP_1$;

$\therefore$ $\angle L_1P_1T = \angle TP_1F$.

Also, $\angle FP_1N = \angle NP_1Q$, since $P_1N \perp P_1T$.

(3) *Through any point in the plane two tangents can be drawn to the parabola* (cf. Arts. 89, 125).

The line
$$y = mx + \frac{p}{m} \qquad (1)$$
is tangent to the parabola $y^2 = 4px$ for all values of m. If $P' \equiv (x', y')$ be any given point of the plane, then the tangent (1) will pass through P' if, and only if, m satisfy the equation
$$y' = mx' + \frac{p}{m},$$
i.e., if
$$m = \frac{y' \pm \sqrt{y'^2 - 4px'}}{2x'}. \qquad (2)$$

Therefore two, and only two, values of m satisfy the given conditions; and therefore through any point of the plane two tangents can be drawn to the parabola. If, however, P' is on the curve, then $y'^2 - 4px' = 0$, the two values of m are equal, *i.e.*, the two tangents coincide. If P' is inside the parabola, then $y'^2 - 4px' < 0$, and the two values of m are imaginary, *i.e.*, there are no real tangent lines. Therefore it is only when P' is outside the parabola that two real and different tangent lines may be drawn from it to the parabola.

(4) *Through any point in the plane three normals can be drawn to the parabola.*

The line
$$y = mx - 2pm - pm^3 \quad . \quad . \quad . \quad (1)$$
is normal to the parabola $y^2 = 4px$ for all values of m (Art. 136). If $P' \equiv (x', y')$ be any point of the plane, then the normal (1) will pass through P' if, and only if, m has a value that will satisfy the equation
$$y' = x'm - 2pm - pm^3. \quad . \quad . \quad . \quad (2)$$
Since equation (2) is a cubic in m, there are three values of m which satisfy the given conditions, and therefore, in general, three normals may be drawn to a parabola from a given point. Special cases may, however, arise in which two of the roots of equation (2) are equal, when there would be only two different normal lines; or all the roots may be equal,* or two imaginary and one real, in both of which cases there would be only one normal line. Through every point at least one normal line can be drawn to the parabola.

(5) *The tangents at the extremities of a focal chord intersect on the directrix, and at right angles* (cf. (6), below).

For, if $S \equiv (x', y')$ is the point of intersection of the tangents at the extremities of the focal chord, then the chord is the polar of S, and its equation is
$$y'y = 2p(x + x'). \quad . \quad . \quad . \quad (1)$$

---

* For only one point, viz.: $P' \equiv (2p, 0)$, are all the roots of equation (2) equal.

But since this line passes through the focus $F \equiv (p, 0)$,

$$\therefore \qquad 0 = 2p(p + x');$$

i.e.,

$$x' = -p. \qquad \qquad (2)$$

Hence the point P' is on the locus $x = -p$, *i.e.*, on the directrix.

Again, the tangent line

$$y = mx + \frac{p}{m} \qquad \qquad (3)$$

passes through the point $P' \equiv (-p, y')$

if

$$y' = -mp + \frac{p}{m},$$

i.e., if

$$m^2 + \frac{y'}{p} m - 1 = 0. \qquad \qquad (4)$$

But the roots of equation (4) are the slopes m' and m'' of the two tangents at P_1 and P_2; and by Art. 11,

$$m'm'' = -1.$$

Hence, the tangents at P_1 and P_2 intersect at right angles.

(6) *The line joining any point in the directrix to the focus of a parabola is perpendicular to the chord of contact corresponding to that point.*

For

$$\triangle SL_1P_1 = \triangle SFP,$$

since $L_1P_1 = FP_1$, SP_1 is common, $\angle L_1P_1S = \angle SP_1F$;

hence,

$$\angle SFP_1 = \angle SL_1P_1 = 90°.$$

The property of (5) may now be shown geometrically. Draw the tangent at P_2, and suppose it to meet the directrix in S'; then, by what has just been proved, $\angle S'FP_2$ is a right angle; then FS' must coincide with FS; and the tangents at P_1 and P_2 meet on the directrix.

Moreover, $\angle P_2SP_1$ is a right angle, for SP_1 bisects $\angle FSL_1$, and SP_2 bisects $\angle L_2SF$.

(7) *A perpendicular let fall from the focus upon a tangent line meets that tangent upon the tangent at the vertex.*

For the equation of the tangent at P_1 is

$$y_1y = 2px + 2px_1, \qquad (1)$$

and the equation of the perpendicular through the focus $F \equiv (p, 0)$ is

$$2py = -y_1x + py_1. \qquad (2)$$

Regarding equations (1) and (2) as simultaneous, and solving to find the point of intersection R, its abscissa is determined by the equation

$$(4p^2 + y_1^2)x + p(4px_1 - y_1^2) = 0;$$

or, since

$$y_1^2 = 4px_1,$$

$$x = 0; \qquad (3)$$

and R is therefore on the tangent at A.

NOTE. The preceding properties of the parabola have for variety been given in some cases a geometric, in others an analytic, proof. The student is advised to use both methods of proof for each proposition. Other properties of the parabola are given below as exercises for the student, and should be derived by analytic methods.

EXERCISES

1. Write the equations of the normals drawn through the point (3, 3) to the parabola $y^2 = 6x$.

2. The focal distance of any point of the parabola $y^2 = 4px$ is $p + x$.

3. The circle on a focal chord as diameter touches the directrix.

4. The angle between two tangents to a parabola is one half the angle between the focal radii of the points of tangency.

5. The polars of all points on the latus rectum meet the axis of the parabola in the same point; find its coördinates, for the parabola $y^2 = 4px$.

6. The product of the segments of any focal chord of the parabola $y^2 = 4px$ equals p times the length of the chord.

7. Two tangents are drawn from an external point $P_1 \equiv (x_1, y_1)$ to a parabola, and a third is drawn parallel to their chord of contact. The intersections of the third with each of the other two is half way between P_1 and the corresponding point of contact.

8. The area of a triangle formed by three tangents to a parabola is one half the area of the triangle formed by the three points of tangency.

9. The tangent at any point of the parabola will meet the directrix and latus rectum produced, in two points equidistant from the focus.

10. The normal at one extremity of the latus rectum of a parabola is parallel to the tangent at the other extremity.

11. The tangents at the ends of the latus rectum are twice as far from the focus as they are from the vertex.

12. The circle on any focal radius as diameter touches the tangent drawn at the vertex of the parabola.

13. The line joining the focus to the pole of a chord bisects the angle subtended at the focus by the chord.

14. Prove, geometrically, that a perpendicular let fall from the focus upon a tangent line of a parabola meets that tangent upon the tangent drawn at the vertex (cf. (7) of Art. 138, p. 229).

139. Diameters. A diameter has been defined as the locus of the middle points of a system of parallel chords. Its equation may be found as follows (cf. Art. 129):

Let m be the common slope of a system of parallel chords of the parabola whose equation is

$$y^2 = 4px, \quad . \quad . \quad . \quad (1)$$

then the equation of one of these chords is

$$y = mx + k, \quad . \quad . \quad . \quad (2)$$

and the equation of any other chord of the system will differ from this only in the value of the constant term k. The chord (2) meets the parabola (1) in two points

$$P_1 \equiv (x_1, y_1)$$

and $\quad P_2 \equiv (x_2, y_2),$

and the coördinates of the middle point $P' \equiv (x', y')$ are therefore

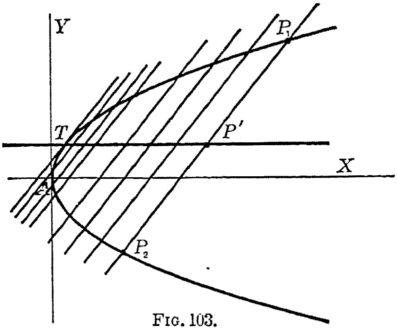


FIG. 103.

$$x' = \frac{x_1 + x_2}{2}, \quad \text{and} \quad y' = \frac{y_1 + y_2}{2}. \quad . \quad . \quad . \quad (3)$$

Considering (1) and (2) as simultaneous equations, and eliminating x, it follows that the ordinates of P_1 and P_2 are the roots of the equation

$$my^2 = 4p(y - k),$$

i.e., of

$$y^2 - \frac{4p}{m}y + \frac{4pk}{m} = 0. \quad . \quad . \quad . \quad (4)$$

Therefore, by Art. 11,

$$y_1 + y_2 = \frac{4p}{m}, \; i.e., \; y' = \frac{2p}{m};$$

hence whatever the value of k, the coördinates of the middle point of the chord satisfy the equation

$$y = \frac{2p}{m}. \quad . \quad . \quad . \quad (5)$$

This is, therefore, the equation of the diameter corresponding to the system of chords whose slope is m.*

---

* Equation (5) might have been obtained at once as a special form of equation [54], Art. 129, by giving appropriate values to the coefficients A, B, F, G, and C there used.

140. Some properties of the parabola involving diameters. The equation of the diameter of the parabola (Art. 139),

$$y = \frac{2p}{m}, \quad . \quad . \quad . \qquad (1)$$

shows at once that *every diameter of the parabola is parallel to the axis of the curve.* (See also Ex. 8, p. 213.)

Conversely, since any value whatever may be assigned to m, each value determining a system of parallel chords, equation (1) may represent *any* line parallel to the x-axis, and therefore *every line parallel to the axis of a parabola bisects some set of parallel chords, and is a diameter of the curve.*

Again, each of the chords cuts the parabola in general in two distinct points, and the nearer these chords are to the extremity of the diameter the nearer are these two points to each other and to their mid-point. In the limiting position, when the chord passes through the extremity of the diameter, the two intersection points and their mid-point become coincident, and the chord is a tangent. Therefore *the tangent at the end of a diameter is parallel to the bisected chords.*

It follows from the preceding properties, or directly from equation (1), that *the axis of the parabola is the only diameter perpendicular to the tangent at its extremity.*

The student will readily perceive how the above properties give a method for constructing a diameter to a set of chords, and in particular how to construct the axis of a given parabola. Thus the problem of Art. 137, to construct a tangent and normal to a given parabola at a given point, can now be solved even when the axis is not given.

If any point on a diameter is taken as a pole, its polar will be one of the system of bisected chords, of slope m.

For the pole is $P' \equiv \left(x', \frac{2p}{m}\right)$, hence the equation of its polar (Art 127) is

$$\frac{2p}{m}y = 2p(x + x'),$$

i.e.,

$$y = mx + mx',$$

which is the equation of a chord of slope m. In other words, *the tangents at the extremities of a chord of the parabola intersect upon the corresponding diameter.*

141. The equation of a parabola referred to any diameter and the tangent at its extremity as axes. In the simplest form of the equation of the parabola, viz.,

$$y^2 = 4px, \qquad . \quad . \quad . \qquad (1)$$

the coördinate axes are the *principal* diameter and the tangent at its extremity. These are the only pair of such lines that are perpendicular to each other (Art. 140). It is now desired to find the equation of the parabola, when referred to any diameter of the curve and the tangent at its extremity as axes.

Let any diameter $O'X'$ of the parabola (1) be the new x-axis, and the tangent $O'Y'$ at O' be the new y-axis, meeting the old x-axis at an angle θ.

If $\qquad m = \tan\theta, \quad . \quad . \quad . \quad (2)$

then (Art. 135) the coördinates of O' are $\frac{p}{m^2}$ and $\frac{2p}{m}$, and the equation for transforming the equation from the old axes to a parallel set through the point O' are (Art. 71),

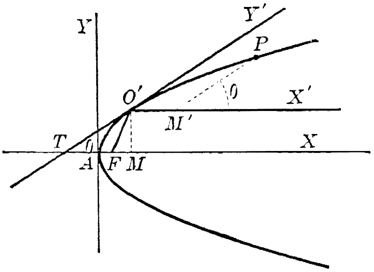


FIG. 104.

$$x = x' + \frac{p}{m^2}, \quad y = y' + \frac{2p}{m}. \qquad . \quad . \quad . \qquad (3)$$

Substituting these values in equation (1) gives

$$y'^2 + \frac{4p}{m}y' = 4px'. \qquad . \quad . \quad . \qquad (4)$$

To turn the y-axis to the final position, making an angle θ with the x-axis, the equations for transformation are (Art. 72, [24]),

$$x' = x'' + y'' \cos\theta,\ y' = y'' \sin\theta,$$

or, by equation (2),

$$x' = x'' + \frac{y''}{\sqrt{1+m^2}}, \quad \text{and} \quad y' = \frac{my''}{\sqrt{1+m^2}}. \quad . \quad . \quad . \qquad (5)$$

Substituting these values in equation (4), it becomes

$$\frac{m^2}{1+m^2}y''^2 = 4px'';$$

or, dropping now the accents,

$$y^2 = 4p\left(\frac{1+m^2}{m^2}\right)x, \quad . \quad . \quad . \quad . \qquad (6)$$

which is the required equation of the parabola.

This equation may, however, be written more simply. Observing (Art. 103) that $p\left(\frac{1+m^2}{m^2}\right)$ is the focal distance of the new origin O', and representing that distance by p', equation (6) becomes

$$y^2 = 4p'x. \quad . \quad . \quad . \qquad [58]$$

This equation is of the same form as equation (1), but is referred to oblique axes. In general, therefore, the equation

$$y^2 = kx$$

represents a parabola, and $\frac{k}{4}$ is the distance of its focus from the origin.

Equation [58] states the following property for every point P of the parabola:

$$\overline{M'P}^2 = 4\,FO' \cdot O'M\ ;$$

a property entirely analogous to that of Art. 106.

EXERCISES

1. Find the diameter of $y^2 = -7x$, which bisects the chords parallel to the line $x - y + 2 = 0$.

2. A diameter of the parabola $y^2 = 8x$ passes through the point $(2, -3)$; what is the equation of its corresponding chords?

3. Find the equation of the diameter of the parabola $y^2 = 4x + 4$ which bisects the chords $2y - 3x = k$.

4. Find the equation of the tangent to the parabola $(y-6)^2 = 8(x+2)$, which is perpendicular to the diameter $y - 4 = 0$.

5. Show that the pole of any chord is on the diameter which corresponds to the chord.

6. What is the equation of the parabola $y^2 = 8x$, when referred to its diameter $y - 5 = 0$ and the corresponding tangent as coördinate axes?

7. What is the equation of the parabola $(x + 3)^2 = 12(y - 1)$, when referred to a diameter through the point $(3, 4)$ and the corresponding tangent as coördinate axes?

8. Find the pole of the diameter $y = k$ with reference to the parabola $y^2 = 4px$.

9. The polar of any point on a diameter is parallel to the corresponding tangent of that diameter.

EXAMPLES ON CHAPTER IX

Find the equation of a parabola with axis parallel to the x-axis:

1. passing through the points $(0, 0)$, $(3, 2)$, $(3, -2)$;

2. passing through the points $(1, 1)$, $(-3, -3)$, $(2, 2)$;

3. through the point $(4, -5)$, with the vertex at the point $(3, -7)$.

4. Find the equation of a parabola through the four points $(0, 2)$, $(3, 0)$, $(-1, -1)$, $(-3, -2)$.

5. Find the vertex and axis of the parabola of Ex. 4.

Find the equation of a parabola

6. if the axis and directrix are taken as coördinate axes.

7. with the focus at the origin, and the y-axis parallel to the directrix.

8. tangent to the line $4y = 3x - 12$, the equation being in the simplest standard form.

9. if a focal radius of length 10 lies along the line $4x - 3y - 8 = 0$.

10. Two equal parabolas have the same vertex, and their axes are perpendicular; find their common chord and common tangent (cf. Ex. 10, p. 224).

11. At what angle do the parabolas of Ex. 10 intersect?

12. Two tangents to a parabola are perpendicular to each other; find the product of the corresponding sub-tangents.

Find the locus of the middle point

13. of all the ordinates of a parabola.

14. of all chords passing through the vertex.

15. From any point on the latus rectum of a parabola, perpendiculars are drawn to the tangents at its extremities; show that the line joining the feet of these perpendiculars is a tangent to the parabola.

16. If tangents are drawn to the parabola $y^2 = 4\,ax$ from any point on the line $x + 4\,a = 0$, their chord of contact will subtend a right angle at the vertex.

Two tangents of slope m and m', respectively, are drawn to a parabola; find the locus of their intersection:

17. if $mm' = k$;

18. if $\frac{1}{m} + \frac{1}{m'} = k$;

19. if $\frac{1}{m} - \frac{1}{m'} = k$.

20. Find the locus of the center of a circle which passes through a given point, and touches a given line.

21. The latus rectum of the parabola is a third proportional to any abscissa and the corresponding ordinate.

22. Find the locus of the point of intersection of tangents drawn at points whose ordinates are in a constant ratio.

23. What is the equation of the chord of the parabola $y^2 = 3\,x$ whose middle point is at $(2, {}^{-}5)$?

24. A double ordinate of the parabola $y^2 = 4px$ is $8p$; prove that the lines from the vertex to its two ends are perpendicular to each other.

25. Find the locus of the center of a circle which is tangent to a given circle and also to a given straight line.

26. Find the intersections of a normal to the parabola with the curve, and the length of the intercepted portion.

27. Prove that the locus of the middle point of the normal intercepted between the parabola and its axis is a parabola whose vertex is the focus, and whose latus rectum is one fourth that of the original parabola.

28. Prove that two confocal parabolas, with their axes in opposite directions, intersect at right angles.

29. Find the equation of the parabola when referred to tangents at the extremities of the latus rectum as coördinate axes.

30. The product of the tangent and normal lengths for a certain point of the parabola $y^2 = 4px$ is twice the square of the corresponding ordinate; find the point and the slope of the tangent line.

CHAPTER X

THE ELLIPSE, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

142. Review. In Chapter VIII the nature of the ellipse has been briefly discussed, and its equation found in the two standard forms:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

when the axes of the curve are coincident with the coördinate axes; and

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1,$$

when the axes of the curve are parallel to the coördinate axes, and the center is the point (h, k). In the present chapter it is desired to study some of the intrinsic properties of the ellipse, *i.e.*, properties which belong to the curve but are independent of the coördinate axes; and these can for the most part be obtained most easily from the simpler equation,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ has its eccentricity given by the relation $b^2=a^2(1-e^2)$, *i.e.*, $e^2=\frac{a^2-b^2}{a^2}$; its foci are the two points $(\pm ae, 0)$, and its directrices the lines $x=\pm\frac{a}{e}$ (Art. 110). If the axes are equal, so that $b=a$, the curve takes the special form of the circle, with eccentricity $e=0$, the two foci coincident at the center, and the directrices infinitely distant.

The equation $\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=1$ represents the polar of the point (x_1, y_1) with respect to the ellipse; if the point is outside the curve, this polar line is its chord of contact; if upon the curve, the polar is the tangent at that point (Arts. 122, 126, 127).

These facts will be assumed in the following work.

143. The equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in terms of its slope. The equation of a line having the given slope m is

$$y = mx + k\,; \quad . \quad . \quad . \quad (1)$$

it is desired to find that value of k for which this line will become tangent to the ellipse whose equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \quad . \quad . \quad . \quad (2)$$

Considering equations (1) and (2) as simultaneous, and eliminating y, the resulting equation

$$(b^2+a^2m^2)x^2+2\,a^2mkx+a^2k^2-a^2b^2=0 \quad . \quad . \quad (3)$$

determines the abscissas of the two points of intersection of the curves (1) and (2). When the curves are tangent, these abscissas are equal; therefore

$$a^4m^2k^2-(b^2+a^2m^2)(a^2k^2-a^2b^2)=0,$$

i.e., $$k^2=a^2m^2+b^2,$$

and $$k=\pm\sqrt{a^2m^2+b^2}.$$

Hence $$y=mx\pm\sqrt{a^2m^2+b^2} \quad . \quad . \quad . \quad [59]$$

is the equation of a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, for all values of m.

Equation [59] shows that there are *two* tangents to an ellipse parallel to any given line; and also (Art. 125), that there are two tangents to an ellipse from any external point.

144. The sum of the focal distances of any point on an ellipse is constant; it is equal to the major axis.

The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ has its foci at the points

$$F_1 \equiv (-ae,\ 0) \text{ and } F_2 \equiv (ae,\ 0);$$

with $b^2 = a^2 - a^2e^2$. (Cf. Art. 110.)

Let $P_1 \equiv (x_1,\ y_1)$ be any point on the curve, so that

$$y_1^2 = b^2 - \frac{b^2x_1^2}{a^2}.$$

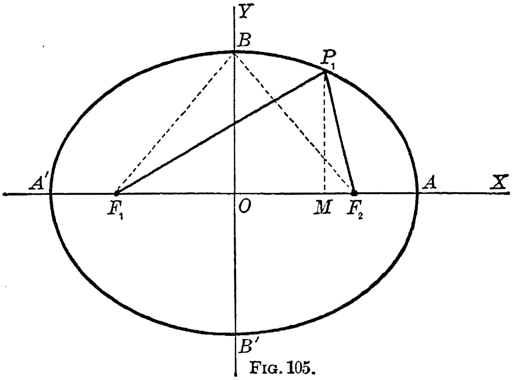


FIG. 105.

Then, $\overline{F_1P_1}^2 = (x_1+ae)^2 + y_1^2 = a^2e^2 + 2\,aex_1 + x_1^2 + y_1^2$

$$= a^2e^2 + 2\,aex_1 + x_1^2 + b^2 - \frac{b^2x_1^2}{a^2}$$

$$= a^2e^2 + 2\,aex_1 + \frac{(a^2-b^2)}{a^2}x_1^2 + a^2 - a^2e^2$$

$$= a^2 + 2\,aex_1 + e^2x_1^2;$$

i.e., $F_1P_1 = a + ex_1.$

Again, $\overline{F_2P_1}^2 = (x_1 - ae)^2 + y_1^2 = a^2e^2 - 2\,aex_1 + x_1^2 + y_1^2$
$= a^2 - 2\,aex_1 + e^2x_1^2,$

i.e., $F_2P_1 = a - ex_1.$

Hence, by addition,

$$F_1P_1 + F_2P_1 = 2\,a\,;$$

i.e., *the sum of the focal distances of any point on an ellipse is constant; it is equal to the major axis.*

This property gives an easy method of finding the foci of an ellipse when the axes $A'A$ and $B'B$ are given.

For $F_1B + F_2B = 2\,a\,;$

but $F_1O = OF_2,$

$\therefore$ $F_2B = F_1B = a.$

Hence, to find the foci, describe arcs with B as center and $a = OA$ as radius, cutting $A'A$ in the points F_1 and F_2; these points are the required foci.

145. Construction of the ellipse. The property of Art. 144 is sometimes given as the definition of the ellipse; viz. *the ellipse is the locus of a point the sum of whose distances from two fixed points is constant.* This definition leads at once to the equation of the curve (cf. Ex. 5, p. 67); and also gives a ready method for its construction.

(*a*) *Construction by separate points.* Let $A'A$ be the given sum of the focal distances, *i.e.*, the major axis of the ellipse; and F_1 and F_2 be the given fixed points, the foci. With either focus as center, and with any radius $A'R < A'A$ describe an arc; then with the other focus as center, and radius RA, describe an arc cutting the first arc in two points. These are points of the ellipse. In the same way as many points as desired may be constructed; a smooth curve connecting these points is approximately an ellipse.

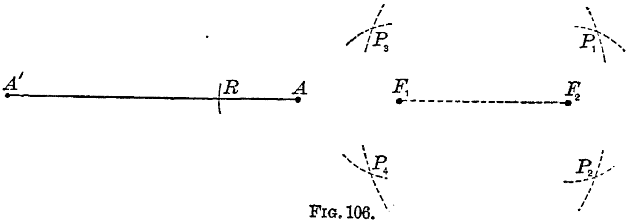


FIG. 106.

(β) *Construction by a continuously moving point.* Fix two upright pins at the foci, and over them place a loop of string, equal in length to the major axis plus the distance between the foci. Press a pencil point against the chord so as to keep it taut. As the pencil moves around the foci, it will trace an ellipse.

EXERCISES

1. Construct an ellipse with semi-axes 8^{cm} and 6^{cm}.

2. Construct an ellipse with semi-axes 5^{cm} and 12^{cm}.

3. Construct an ellipse with the distance between the foci 24, and the minor axis of length 10.

4. Write the equation of the polar of the left-hand focus of the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$. What line is this?

5. By employing equation [59], find the tangent to the ellipse $16x^2+25y^2=400$, and passing through the point (3, 4).

6. By the method of Ex. 17, p. 225, show that an ellipse is concave toward its center.

7. Through what point of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ must a tangent and normal be drawn, to form with the x-axis an isosceles triangle?

8. Write the equations of the tangent and normal at the positive end of the latus rectum of the ellipse $x^2+4y^2=4$. Where do these lines cut the x-axis?

9. Tangents to the ellipse $4x^2 + 3y^2 = 5$ are inclined at 60° to the x-axis; find the points of contact.

10. Find the equation of an ellipse (center at the origin) of eccentricity $\frac{4}{5}$, such that the subtangent for the point $(3, \frac{12}{5})$ is $(-\frac{16}{3})$.

11. Find the chord of contact for tangents from the point $(3, 2)$ to the ellipse $x^2 + 4y^2 = 4$. Find also the equation of the line from $(3, 2)$ to the middle point of this chord.

12. Find the tangents to the ellipse $7x^2 + 8y^2 = 56$ which make the angle $\tan^{-1} 3$ with the line $x + y + 1 = 0$.

13. Find the product of the two segments into which a focal chord is divided by the focus of an ellipse.

14. Find the equation of a tangent, and also of a normal, to the ellipse $x^2 + 4y^2 = 16$, each parallel to the line $3x - 4y = 5$.

15. Find the pole of the line $3x - 4y = 5$ with reference to the ellipse $x^2 + 4y^2 = 16$; also the intercepts on the axes made by a line through the pole and perpendicular to the polar.

16. Find the points on the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, such that the tangent makes equal (numerical) angles with the axes; such that the subtangent equals the subnormal.

146. Auxiliary circles. Eccentric angle. The circumscribed and inscribed circles for the ellipse (Fig. 107) are called **auxiliary** circles, and bear an important part in the theory of the ellipse. Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad . \quad . \quad . \quad (1)$$

The circle described on its major axis as diameter is called the **major auxiliary circle**; its equation is

$$x^2 + y^2 = a^2; \quad . \quad . \quad . \quad (2)$$

and the circle on the minor axis as diameter is the **minor auxiliary circle**; its equation is

$$x^2 + y^2 = b^2. \quad . \quad . \quad . \quad (3)$$

If $\angle AOQ$ is any angle ϕ at the center of the ellipse, with the initial side on the major axis, and the terminal side cutting the auxiliary circles in R and Q, respectively; and if P is the intersection of the abscissa LR with the ordinate MQ, then P is a point on the ellipse.

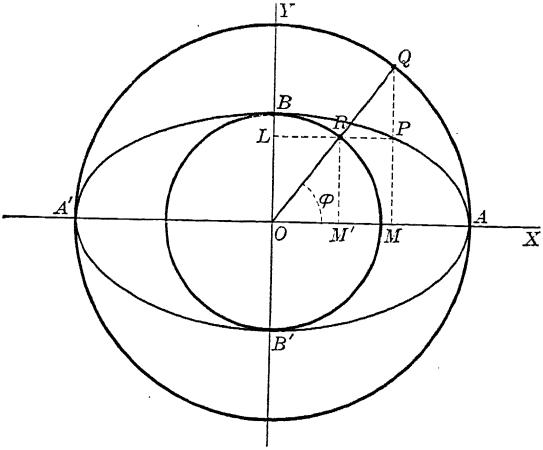


FIG. 107.

For the coördinates of P are

$$OM = OQ \cos\phi \text{ and } MP = M'R = OR \sin\phi,$$

i.e., $$x = a\cos\phi, \qquad y = b\sin\phi. \quad . \quad . \quad . \qquad [60]$$

Now these values satisfy the equation of the ellipse; for, substituting them in equation (1), gives

$$\frac{a^2\cos^2\phi}{a^2} + \frac{b^2\sin^2\phi}{b^2} = \cos^2\phi + \sin^2\phi = 1;$$

hence P is a point of the ellipse.

The points P, Q, and R are called **corresponding points.** The angle ϕ is the **eccentric angle** of the point P;* and the two equations [60] are equations of the ellipse in terms of the eccentric angle, for together they express the condition that the point P is on the ellipse (1).*

---

* The eccentric angle of any given point P on an ellipse is readily constructed thus: produce the ordinate MP to meet the major auxiliary circle in Q; the angle AOQ is the eccentric angle of the point P.

Since, in the figure, △ $OM'R$ and OMQ are similar, it follows that

$$MP : MQ = OR : OQ = b : a,$$

and $$OM' : OM = OR : OQ = b : a;$$

that is, *the ordinate of any point on the ellipse is to the ordinate of the corresponding point on the major auxiliary circle in the ratio* $(b : a)$ *of the semi-axes.* Similarly for the abscissas of the corresponding points R and P.

147. The subtangent and subnormal. Construction of tangent and normal.

Let $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad . \quad . \quad . \qquad (1)$$

be a given ellipse,

then $$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1, \quad . \quad . \quad . \qquad (2)$$

is the tangent to it at a point $P_1 \equiv (x_1, y_1)$. Let this tangent cut the x-axis at the point T. Draw the ordinate MP_1.

Then the subtangent is, by definition, TM; and its numerical value is

$$MT = OT - OM;$$

but, from equation (2), $OT = \frac{a^2}{x_1}$; and $OM = x_1$;

hence $$MT = \frac{a^2}{x_1} - x_1,$$

i.e., $$TM = \frac{x_1^2 - a^2}{x_1}.$$

---

* The equations [60] are of great service in studying the ellipse by the methods of the differential calculus.

Hence the value of the subtangent, corresponding to any point of the ellipse whose equation is (1), depends only upon the major axis, and the abscissa of the point; therefore, if *a series of ellipses have the same major axis, tangents drawn to them at the points having a common abscissa will cut the major axis (extended) in a common point.*

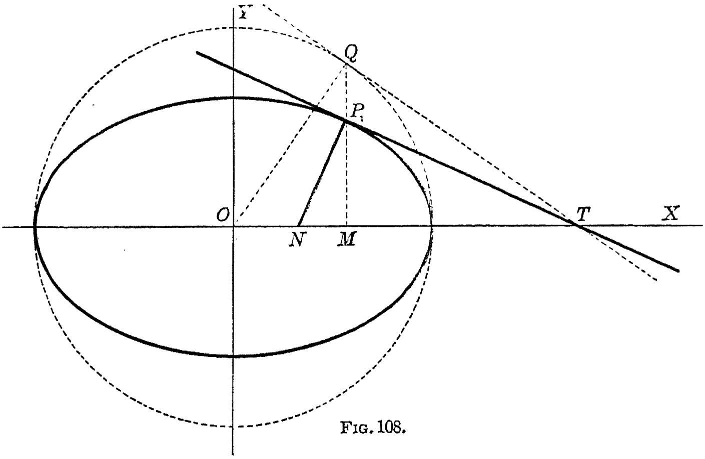


FIG. 108.

This fact suggests a method for constructing a tangent and normal to an ellipse, at a given point: draw the major auxiliary circle; at Q on this circle, and in MP_1 extended, draw a tangent to the circle. This will cut the axis in T; and P_1T will be the required tangent of the ellipse at P_1. The normal P_1N may then be drawn perpendicular to P_1T.

The equation of the normal through P_1 is (cf. eq. [51])

$$y - y_1 = \frac{a^2 y_1}{b^2 x_1}(x - x_1);$$

therefore the x-intercept of the normal at that point is

$$ON = \frac{a^2 - b^2}{a^2} x_1 = e^2 x_1.$$

But the subnormal corresponding to P_1 is

$$MN = ON - OM,$$

and $$OM = x_1;$$

therefore $$MN = \frac{a^2 - b^2}{a^2}x_1 - x_1$$

$$= -\frac{b^2}{a_2}x_1 = (e^2 - 1)x_1.$$

NOTE. From the value of ON it follows that the normal to an ellipse does not, in general, pass through the center, but passes between the center and the foot of the ordinate; the extremities of the axes of the curve being exceptional points. If, however, $a = b$, then $e = 0$, the curve is a circle, and every normal passes through the center (cf. Art. 85).

148. The tangent and normal bisect externally and internally, respectively, the angles between the focal radii of the point of contact.

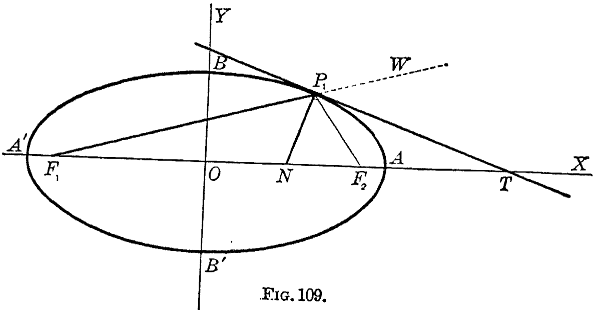


FIG. 109.

Let the equation of the given ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; also let F_1 and F_2 be the foci, and $P_1 \equiv (x_1, y_1)$ any given point on the curve. Draw the tangent TP_1, the normal P_1N, and also the lines F_2P_1 and F_1P_1W.

Then $F_1N = F_1O + ON = ae + e^2x_1$ [Art. 147]

$$= e(a + ex_1),$$

$$NF_2 = OF_2 - ON = ae - e^2x_1$$

$$= e(a - ex_1),$$

also $F_1P_1 = a + ex_1$, [Art. 144]

and $F_2P_1 = a - ex_1$.

Hence $F_1N : NF_2 = F_1P_1 : P_1F_2$;

and, by a theorem of plane geometry, this proportion proves that the normal P_1N bisects the angle $F_1P_1F_2$ between the focal radii. Again, since the tangent is perpendicular to the normal, the tangent P_1T will bisect the external angle F_2P_1W.

This proposition leads to a second method of constructing the tangent and normal to an ellipse at a given point (cf. Art. 147). First determine the foci, F_1 and F_2 (Art. 144), then draw the focal radii to the given point and bisect the angle thus formed,—internally for the normal, externally for the tangent.

149. The intersection of the tangents at the extremity of a focal chord. If $P' \equiv (x', y')$ be the intersection of two tangents to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

the equation of their chord of contact is (Art. 126)

$$\frac{x'x}{a^2} + \frac{y'y}{b^2} = 1. \qquad (1)$$

If this chord passes through the focus $F_2 \equiv (ae, 0)$, its equation must be satisfied by the coördinates of F_2; therefore

$$\frac{x'ae}{a^2} = 1, \quad i.e., x' = \frac{a}{e},$$

and the point of intersection P' is on the line, $x=\frac{a}{e}$; *i.e.*, on the directrix corresponding to the focus F_2. Similarly, if the chord passes through the focus $F_1 \equiv (-ae, 0)$, the point P' is on the directrix $x=-\frac{a}{e}$.

Hence, *the tangents at the extremities of a focal chord intersect upon the corresponding directrix.*

Again, the line joining the intersection $P' \equiv \left(\frac{a}{e}, y'\right)$ to the focus has the slope

$$m' = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y'}{\frac{a}{e} - ae} = \frac{ey'}{a(1-e^2)} = \frac{aey'}{b^2};$$

while the slope of the focal chord (1) is

$$m = -\frac{b^2x'}{a^2y'} = -\frac{b^2}{aey'};$$

hence

$$m' = -\frac{1}{m},$$

and therefore *the line joining the focus to the intersection of the tangents at the ends of a focal chord is perpendicular to that chord.*

150. The locus of the foot of the perpendicular from a focus upon a tangent to an ellipse. Let the equation of a tangent to the ellipse (Art. 143), whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad (1)$$

be written in the form $\quad y = mx + \sqrt{a^2m^2 + b^2}. \qquad (2)$

Then the equation of a perpendicular to (2), through the focus $(ae, 0)$, is

$$y = -\frac{1}{m}(x - ae), \quad i.e.,\ x + my = ae. \qquad (3)$$

If $P' \equiv (x', y')$ is the point of intersection of (2) and (3), it is required to find the locus of P'; *i.e.*, to find an equation which will be satisfied by the coördinates x', y', whatever the value of m; this must be an equation involving x' and y', but free from m. Since P' is on both lines (2) and (3),

therefore $\quad y' - mx' = \sqrt{a^2m^2 + b^2}, \qquad (4)$

and $\quad x' + my' = ae. \qquad (5)$

The elimination of m is accomplished most easily by squaring each member of equations (4) and (5), and adding:

this gives $$(1 + m^2)\,x'^2 + (1 + m^2)\,y'^2 = a^2m^2 + a^2e^2 + b^2,$$

i.e., $$(1 + m^2)(x'^2 + y'^2) = a^2\,(m^2 + 1),$$

whence, $$x'^2 + y'^2 = a^2.$$

Hence, the point P' is on the circle

$$x^2 + y^2 = a^2;$$

that is, *the locus of the foot of a perpendicular from either focus upon a tangent to the ellipse is the major auxiliary circle.*

151. **The locus of the intersection of two perpendicular tangents to the ellipse.**

Let the equation of any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be written in the form (Art. 143)

$$y - mx = \sqrt{a^2m^2 + b^2}, \quad . \quad . \quad . \quad (1)$$

then the equation of a perpendicular tangent is

$$y + \frac{1}{m}\,x = \sqrt{\frac{a^2}{m^2} + b^2},$$

i.e., $$my + x = \sqrt{a^2 + b^2m^2}. \quad . \quad . \quad . \quad (2)$$

Letting $P' \equiv (x', y')$ be the point of intersection of these two tangents, (1) and (2), it is required to find the locus of P' as m varies in value: that is, to find an equation between x' and y' which does not involve m.

Proceeding as in Art. 150; since P' is on both lines (1) and (2),

therefore $$y' - mx' = \sqrt{a^2m^2 + b^2},$$

and $$my' + x' = \sqrt{a^2 + b^2m^2}.$$

To eliminate m, square both equations, and add: this gives

$$(m^2 + 1)\,y'^2 + (m^2 + 1)\,x'^2 = (m^2 + 1)\,a^2 + (m^2 + 1)\,b^2,$$

i.e., $$x'^2 + y'^2 = a^2 + b^2.$$

Therefore, the point of intersection of perpendicular tangents is on the circle

$$x^2 + y^2 = a^2 + b^2, \quad . \quad . \quad . \quad [61]$$

which is called the **director circle** for the ellipse. *The locus of the intersection of two perpendicular tangents to an ellipse is, then, its director circle.*

EXERCISES

1. Prove that the two tangents drawn to an ellipse from any external point subtend equal angles at the focus.

2. Each of the two tangents drawn to the ellipse from a point on the directrix subtends a right angle at the focus.

3. A focal chord is perpendicular to the line joining its pole to the focus. Show that this is also true for a parabola.

4. The rectangle formed by the perpendiculars from the foci upon any tangent is constant; it is equal to the square of the semi-minor-axis.

5. The circle on any focal distance as diameter touches the major auxiliary circle.

6. The perpendicular from the focus upon any tangent, and the line joining the center to the point of contact, meet upon the directrix.

7. The perpendicular from either focus, upon the tangent at any point of the major auxiliary circle, equals the distance of the corresponding point of the ellipse from that focus.

8. The latus rectum is a third proportional to the major and minor axes.

9. The area of the ellipse is πab.

SUGGESTION. Employ the fact, proved in Art. 146, that the ordinate of an ellipse is to the corresponding ordinate of the major auxiliary circle as $b:a$, and thus compare the area of the ellipse with that of its major auxiliary circle.

152. Diameters. As already shown in Articles 129 and 139, the definition of a diameter as the locus of the middle points of a system of parallel chords leads directly to its equation.

Let m be the slope of the given system of parallel chords of the ellipse whose equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \qquad (1)$$

and let

$$y=mx+c \qquad (2)$$

be the equation of one of these chords, which meets the curve in the two points $P_1 \equiv (x_1, y_1)$ and $P_2 \equiv (x_2, y_2)$. Let $P' \equiv (x', y')$, be the middle point of this chord, so that

$$x' = \frac{x_1 + x_2}{2}, \quad y' = \frac{y_1 + y_2}{2}. \quad . \quad . \quad . \quad (3)$$

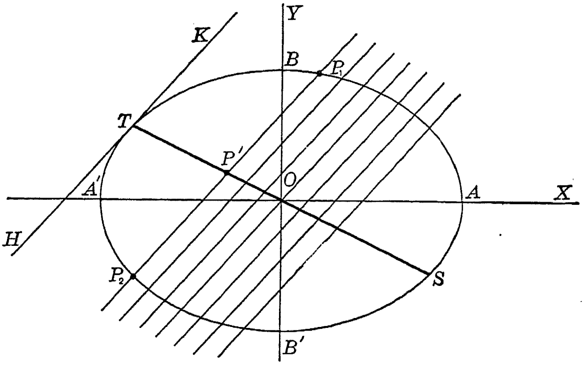


FIG. 110

The coördinates of P_1 and P_2 are found by solving (1) and (2) as simultaneous equations, therefore the abscissas x_1 and x_2 are the roots of the equation

$$(a^2m^2 + b^2)x^2 + 2\,a^2cmx + a^2c^2 - a^2b^2 = 0, \quad . \quad . \quad . \quad (4)$$

and the ordinates y_1 and y_2 are roots of the equation

$$(a^2m^2 + b^2)y^2 - 2\,b^2cy + b^2c^2 - a^2b^2m^2 = 0. \quad . \quad . \quad . \quad (5)$$

Hence, by Art. 11, the coördinates of P' are

$$x' = -\frac{a^2cm}{a^2m^2 + b^2}, \quad y' = \frac{b^2c}{a^2m^2 + b^2}. \quad . \quad . \quad . \quad (6)$$

Now, by varying the value of c, equation (6) gives the coördinates of the middle point for each of the chords of the given set. It is required to find the locus of P' for all values of c, *i.e.*, to find an equation satisfied by x' and y', and not dependent upon the value of c. If x' be divided by y', the c is eliminated from the equations (6), giving

$$\frac{x'}{y'} = -\frac{a^2}{b^2}m. \quad . \quad . \quad . \qquad (7)$$

Therefore the coördinates of the middle point of every chord of slope m satisfy the equation

$$\frac{x}{y} = -\frac{a^2}{b^2}m,$$

or,
$$y = -\frac{b^2}{a^2m}x; \quad . \quad . \quad . \qquad [62]$$

which is therefore the equation of the diameter bisecting the chords of slope m.

The form of equation [62] shows that *every diameter of the ellipse passes through the center.*

153. Conjugate diameters. Since every diameter passes through the center of the ellipse, and since, by varying the slope m of the given set of parallel chords, the corresponding diameter may be made to have any required slope, therefore it follows that *every chord which passes through the center of an ellipse is a diameter*, corresponding to some set of parallel chords. In particular, that one of the set of chords given by equation (2), Art. 152, which passes through the center, —*i.e.*, the chord whose equation is

$$y = mx, \quad . \quad . \quad . \qquad [63]$$

is a diameter. This diameter bisects the chords parallel to the line [62]; for if m' be the slope of the line [62],

then
$$m' = -\frac{b^2}{a^2m},$$

hence,
$$mm' = -\frac{b^2}{a^2}; \quad . \quad . \quad . \qquad [64]$$

and this equation expresses the condition that line [62], which has the slope m', shall bisect the chords of slope m (Art. 152). But conversely, it expresses also the condition that the line [63] which has the slope m shall bisect the chords of slope m'. Hence each of the lines [62] and [63] bisects the chords parallel to the other. Hence, *if one diameter bisects the chords parallel to a second, then also the second diameter bisects the chords parallel to the first.* Such diameters are called **conjugate** to each other.

Each line of the set of parallel chords in general cuts the ellipse in two distinct points, and the further the chord is from the center, the nearer these two points are to each other, and to their mid-point. In the limiting position, the chord becomes a tangent, with the two intersection points and their mid-point coincident at the point of tangency. Therefore, *the tangent at the end of a diameter is parallel to the conjugate diameter.* This property, with that of Art. 152, suggests a method for constructing conjugate diameters: first draw a tangent at an extremity of a given diameter (Art. 147), then a line drawn parallel to this tangent through the center of the ellipse is the required conjugate diameter. (See Fig. 111.)

154. **Given an extremity of a diameter, to find the extremity of its conjugate diameter.**

Let $P_1 \equiv (x_1, y_1)$ be an extremity of a given diameter (Fig. 111), then $P_2 \equiv (-x_1, -y_1)$ will be the other extremity. Let $P_1' \equiv (x_1', y_1')$ and $P_2' \equiv (-x_1', -y_1')$ be the extremities of the conjugate diameter. Let the equation of the e[illegible] be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1; \quad . \quad . \quad . \quad (1)$$

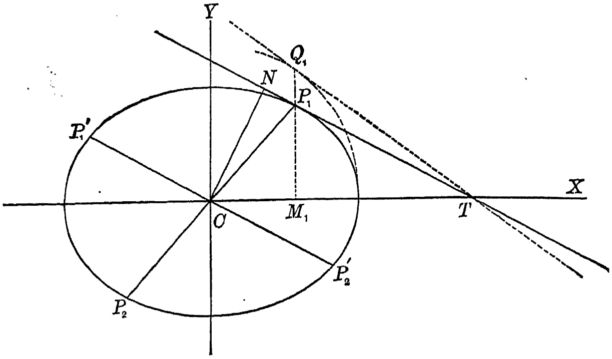
then the equati[illegible] of the given diameter P_1P_2 is

s e

$$y = \frac{y_1}{x_1}x, \quad . \quad . \quad . \quad (2)$$

and that of the conjugate diameter $P_1'P_2'$, through the center and parallel to the tangent at P_1 is

$$\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=0. \quad . \quad . \quad . \qquad (3)$$



The coördinates of P_1' and P_2', in terms of x_1, y_1, a, and b, are given by equations (1) and (3), considered as simultaneous; hence, eliminating y between these equations, and remembering that the point P_1 is on the ellipse (1) and that therefore $b^2x_1^2+a^2y^2=a^2b^2$, the abscissas of the points P_1' and P_2' are given by the equation

$$x^2=\frac{a^2y_1^2}{b^2};$$

i.e.,
$$x_1'=-\frac{a}{b}y_1 \text{ and } x_2'=\frac{a}{b}y_1.$$

Substituting these values in equation (3), gives for the corresponding ordinates,

$$y_1'=\frac{b}{a}x_1 \text{ and } y_2'=-\frac{b}{a}x_1.$$

Therefore the required extremities of the conjugate diameter are

$$P_1'\equiv\left(-\frac{a}{b}y_1,\ \frac{b}{a}x_1\right) \text{ and } P_2'\equiv\left(\frac{a}{b}y_1,\ -\frac{b}{a}x_1\right).$$

155. Properties of conjugate diameter[illegible]f the ellipse. (*a*) It has been seen (Art. 153) that tw[illegible] diameters are conjugate when their slopes satisfy the rel[illegible]n

$$mm'=-\frac{b^2}{a^2}. \quad . \quad . \quad . \qquad (1)$$

It follows, since the product of their slopes is negative, that with the exception of the case where one diameter is the minor axis itself, *conjugate diameters do not both lie in the same quadrant formed by the axes of the curve.*

(β) From the definition (Art. 153) it is evident that the minor and major axes of the ellipse are a pair of conjugate diameters, and they are at right angles to each other. Perpendicular lines, however, in general, fulfill the condition

$$mm' = -1; \qquad . \quad . \quad . \qquad (2)$$

hence, in general, equation (2) is not consistent with equation (1) for other values of m and m' than 0 and ∞, — the slopes for the axes of the curves. But for $b^2 = a^2$, *i.e.*, for the circle, it is clear that every pair of conjugate diameters satisfy equation (2), and are therefore perpendicular to each other. Hence, *the major and minor axes of the ellipse are the only pair of conjugate diameters that are perpendicular to each other.*

(γ) If, in Fig. 111, the lengths of the conjugate semi-axes be $a' = CP_1$, $b' = CP_1'$, then, since

$$P_1 \equiv (x_1, y_1), \ P_1' \equiv \left(-\frac{a}{b}y_1, \frac{b}{a}x_1\right),$$

$$b^2x_1^2 + a^2y_1^2 = a^2b^2, \qquad a'^2 = x_1^2 + y_1^2,$$

and

$$b'^2 = \frac{a^2y_1^2}{b^2} + \frac{b^2x_1^2}{a^2};$$

therefore

$$a'^2 + b'^2 = \frac{b^2x_1^2 + a^2y_1^2}{b^2} + \frac{a^2y_1^2 + b^2x_1^2}{a^2}$$

$$= a^2 + b^2; \qquad . \quad . \quad . \qquad (3)$$

i.e., *the sum of the squares of two conjugate semi-diameters is constant; it is equal to the sum of the squares of the two semi-axes.*

(δ) Referring again to Fig. 111, where CN is perpendicular to the tangent at P_1, the conjugate diameters P_1P_2 and $P_1'P_2'$ intersect at an angle ψ such that

$$\psi = \angle P_1CP_1' = 90^\circ + \angle P_1CN;$$

$$\therefore \qquad \sin\psi = \cos\angle P_1CN = \frac{CN}{CP_1}.$$

But, by Art. 64, since the equation of the tangent at P_1 is

$$b^2x_1x + a^2y_1y = a^2b^2,$$

$$CN = \frac{a^2b^2}{\sqrt{b^4{x_1}^2 + a^4{y_1}^2}} = \frac{ab}{\sqrt{\frac{a^2{y_1}^2}{b^2} + \frac{b^2{x_1}^2}{a^2}}} = \frac{ab}{b'};$$

but

$$CP_1 = a',$$

hence

$$\sin\psi = \frac{ab}{a'b'}, \qquad . \quad . \quad . \qquad (4)$$

and *the angle between two conjugate diameters is* $\sin^{-1}\dfrac{ab}{a'b'}$.

(ϵ) Tangents at the extremities of a pair of conjugate diameters form a parallelogram circumscribed about the ellipse; its sides are parallel to, and equal in length to, the conjugate diameters. Since the area of a parallelogram is equal to the product of its adjacent sides and the sine of the included angle, therefore the area of this circumscribed parallelogram is $4a'b'\sin\psi$, which, by (4), equals $4ab$.

That is, *the area of the parallelogram constructed upon any two conjugate diameters is constant; it is equal to the area of the rectangle upon the axes.*

(ζ) A simple relation exists between the eccentric angles of the extremities of two conjugate diameters.

Let the eccentric angle of $P_1 \equiv (x_1, y_1)$ be ϕ_1 (Fig. 112), and of $P_2 \equiv (x_2, y_2)$ be ϕ_2; then the slopes of the conjugate diameters may be written (cf. Art. 146), for CP_1, $$m = \frac{y_1}{x_1} = \frac{b \sin \phi_1}{a \cos \phi_1},$$

and for CP_2, $$m' = \frac{y_2}{x^2} = \frac{b \sin \phi_2}{a \cos \phi_2}.$$

But $$mm' = -\frac{b^2}{a^2}, \qquad \text{[Art. 155 (a)]}$$

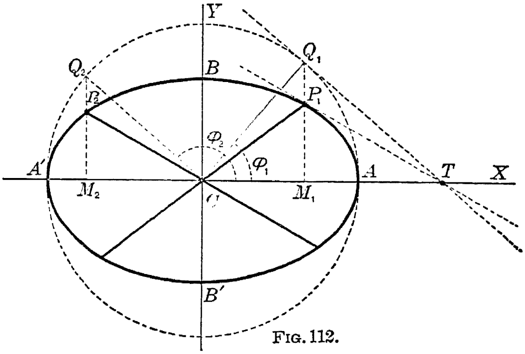


FIG. 112.

hence $$\frac{b^2 \sin \phi_1 \sin \phi_2}{a^2 \cos \phi_1 \cos \phi_2} = -\frac{b^2}{a^2},$$

giving $$\frac{\sin \phi_1 \sin \phi_2}{\cos \phi_1 \cos \phi_2} = -1;$$

that is, $$\sin \phi_2 \sin \phi_1 + \cos \phi_2 \cos \phi_1 = 0,$$

whence $$\cos(\phi_2 - \phi_1) = 0.$$

Therefore $$\phi_2 - \phi_1 = 90°,$$

and *the eccentric angles of the extremities of two conjugate diameters differ by a right angle.*

156. Equi-conjugate diameters. If two conjugate diameters be equal to each other, *e.g.*, if $CP_1 = CP_2$ (see Fig. 112), then the properties given in the preceding article lead to other simple ones.

Let ϕ_1 be the eccentric angle of P_1, then $\phi_1 + 90°$ is the eccentric angle for P_2; hence the coördinates of P_1 and P_2 are $(a \cos \phi_1, b \sin \phi_1)$ and $(-a \sin \phi_1, b \cos \phi_1)$, and since

$$a' = b',$$

therefore $a^2\cos^2\phi_1 + b^2\sin^2\phi_1 = a^2\sin^2\phi_1 + b^2\cos^2\phi_1,$

i.e., $\tan^2\phi_1 = 1.$

Hence $\phi_1 = 45°$ or $135°$

for the extremities of equi-conjugate diameters, and the extremities are

$$P_1 \equiv \left(x_1, \frac{b}{a}x_1\right),\quad P_2 \equiv \left(-x_1, \frac{b}{a}x_1\right).$$

The equations of these diameters are

$$y = \frac{b}{a}x, \text{ and } y = -\frac{b}{a}x.$$

Evidently these lines are the diagonals of the rectangle formed on the axes of the curve.

By Art. 155, (γ), the length of each equi-conjugate semi-diameter is

$$a' = \sqrt{\frac{a^2 + b^2}{2}}.$$

EXERCISES

1. Find the diameter of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ which bisects the chords parallel to the line $3x + 5y + 7 = 0$.

2. Find the diameter conjugate to that of exercise 1.

3. Show that the lines $2x - y = 0$, $x + 3y = 0$ are conjugate diameters of the ellipse $2x^2 + 3y^2 = 4$.

4. For the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, write the equations of diameters conjugate to the line

(α) $ax = by$, (β) $bx = ay$.

5. Prove that the angle between two conjugate diameters is a maximum when they are equal.

6. Show that the pair of diameters drawn parallel to the chords joining the extremities of the axes are equal and conjugate.

7. What are the equations of the pair of equi-conjugate diameters of the ellipse $16y^2 + 9x^2 = 144$?

8. Two conjugate diameters of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ have the slopes $\frac{3}{4}$ and $-\frac{3}{4}$, respectively; find their lengths.

9. Given the ellipse $x^2 + 5y^2 = 5$, find the eccentric angle for the point whose abscissa is 1. Also find the diameter conjugate to the one passing through this point.

10. Given the ellipse $3x^2 + 4y^2 = 12$, find the conjugate diameters for the point whose eccentric angle is 30°.

11. Find the lengths of the diameters in exercise 10.

12. The lengths of the chord joining the extremities of any two conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\sqrt{a^2 + b^2 + a^2e^2 \sin 2\phi}$. Find its greatest value. What is the corresponding value of ϕ?

13. The area of a triangle inscribed in an ellipse, if ϕ_1, ϕ_2, ϕ_3 be the eccentric angles of the vertices, is

$$\tfrac{1}{2} ab\,[\sin(\phi_2 - \phi_3) + \sin(\phi_3 - \phi_1) + \sin(\phi_1 - \phi_2)].$$

14. Given the point $(-3, -1)$ on the ellipse $x^2 + 3y^2 = 12$; find the corresponding point on the major auxiliary circle, and also find the eccentric angle of the given point.

15. Find the polar of the focus of an ellipse with reference to each auxiliary circle.

16. Find the pole of the directrix of the ellipse with reference to each auxiliary circle.

17. Prove analytically that tangents at the ends of any chord intersect on the diameter which bisects that chord.

157. Supplemental chords. The chords drawn from any point of an ellipse to the extremities of a diameter are called **supplemental chords.** Such chords are always parallel to a pair of conjugate diameters, since their slopes satisfy the relation

$$mm' = -\frac{b^2}{a^2}.$$

For if $P_1 \equiv (x_1, y_1)$ and $P_2 \equiv (-x_1, -y_1)$ be the extremities of a diameter, and $P' \equiv (x', y')$ be any other point of the ellipse, and m and m' the slopes of the chords $P'P_1$ and $P'P_2$, respectively,

then $$m = \frac{y' - y_1}{x' - x_1},\quad m' = \frac{y' + y_1}{x' + x_1},$$

therefore $$mm' = \frac{y'^2 - y_1^2}{x'^2 - x_1^2}.$$

But $$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1,$$

and $$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1;$$

hence, by subtraction,

$$\frac{x'^2 - x_1^2}{a^2} + \frac{y'^2 - y_1^2}{b^2} = 0,$$

that is,
$$\frac{y'^2 - y_1^2}{x'^2 - x_1^2} = -\frac{b^2}{a^2};$$

hence
$$mm' = -\frac{b^2}{a^2}.$$

Therefore, *supplemental chords are parallel to a pair of conjugate diameters.*

For the special case when $a = b$, the product of the slopes becomes $mm' = -1$, and therefore the supplemental chords are perpendicular; in other words, the angle inscribed in a semicircle is a right angle.

158. Equation of the ellipse referred to a pair of conjugate diameters.

In the simplest form for the equation of the ellipse, viz.,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad . \quad . \quad . \quad (1)$$

the coördinate axes are the axes of the curve. These axes are conjugate diameters, and they are the only pair which are at right angles to each other (cf. Art. 155, β). It is desired now to find the equation of the curve referred to any pair of conjugate diameters, as P_2P_1 and $P_2'P_1'$, in Fig. 111. With the notation of Art. 154, let θ and θ' be the angles the new x-axis, CP_1, and the new y-axis, CP_1', make with the old x-axis, respectively; they satisfy the relation [64],

$$\tan\theta \tan\theta' = -\frac{b^2}{a^2}. \quad . \quad . \quad . \quad (2)$$

The lengths of the conjugate semi-diameters are $a' = CP_1$ and $b' = CP_1'$.

Then, by Art. 73, the equations for transformation to the new axes are

$$x = x'\cos\theta + y'\cos\theta', \; y = x'\sin\theta + y'\sin\theta', \quad . \quad . \quad . \quad (3)$$

and after transformation equation (1) becomes

$$\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right)x'^2 + 2\left(\frac{\cos\theta\cos\theta'}{a^2} + \frac{\sin\theta\sin\theta'}{b'}\right)x'y'$$
$$+\left(\frac{\cos^2\theta'}{a^2} + \frac{\sin^2\theta'}{b^2}\right)y^3 = 1. \quad . \quad . \quad . \quad (4)$$

But, by (2),
$$\frac{\sin\theta\sin\theta'}{\cos\theta\cos\theta'} = -\frac{b^2}{a^2},$$

hence
$$\frac{\sin\theta\sin\theta'}{b^2}+\frac{\cos\theta\cos\theta'}{a^2}=0,$$

and equation (4) reduces to
$$\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)x'^2+\left(\frac{\cos^2\theta'}{a^2}+\frac{\sin^2\theta'}{b^2}\right)y'^2=1. \quad . \quad . \quad . \quad (5)$$

From equation (5) it is seen that the curve is obliquely symmetrical with respect to the new axes. Moreover, since $\pm a'$ and $\pm b'$ are the intercepts on the new axes, equation (5) may be further simplified:

for
$$\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)a'^2=1,$$

and
$$\left(\frac{\cos^2\theta'}{a^2}+\frac{\sin^2\theta}{b^2}\right)b'^2=1;$$

hence
$$\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}=\frac{1}{a'^2},\quad \frac{\cos^2\theta'}{a^2}+\frac{\sin^2\theta'}{b^2}=\frac{1}{b'^2},$$

and equation (5) may be written
$$\frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1. \quad . \quad . \quad . \quad [65]$$

This is the required equation of the ellipse when referred to any pair of conjugate diameters. It is evident that propositions which were derived for the standard form (1) without reference to the fact that the axes were rectangular, hold equally for equation [65]; *e.g.*, the equation of a tangent at the point (x_1, y_1) of the curve is $\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=1$.

Equation [65] states a geometric property of the ellipse entirely analogous to that of Art. 112. It is left to the student to express this property in words.

If the ellipse is referred to equi-conjugate diameters, so that $a'=b'$, its equation will be
$$x^2+y^2=a'^2. \quad . \quad . \quad . \quad [66]$$

This is the same form as the simplest equation of the circle, but here the axes are oblique, and the equation represents, not a circle, but an ellipse.

159. Ellipse referred to conjugate diameters; second method.

If the ellipse
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad . \quad . \quad . \quad (1)$$

is transformed to a pair of conjugate diameters, its equation after transformation (Art. 73) must be of the form
$$Ax^2+2Hxy+By^2=1. \quad . \quad . \quad . \quad (2)$$

But, since each chord parallel to either axis is bisected by the other, therefore, if (x_1, y_1) is a point on the curve, then $(-x_1, +y_1)$ must also be on the curve;

i.e., if $$Ax_1^2 + 2Hx_1y_1 + By_1^2 = 1,$$

then $$Ax_1^2 - 2Hx_1y_1 + By_1^2 = 1,$$

and, consequently, $$H = 0.$$

Again, $(a', 0)$ and $(0, b')$ are points on the curve;

hence $$Aa'^2 = 1, \qquad Bb'^2 = 1;$$

i.e., $$A = \frac{1}{a'^2}, \qquad B = \frac{1}{b'^2},$$

therefore, equation (2) becomes

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1.$$

This method illustrates how analytic reasoning may often be used to shorten or perhaps obviate the algebraic reductions involved in a proof. With the similar methods of Arts. 39 and 40, it will suggest to the reader the power and interest of what are called the *modern methods* in analytic geometry.

EXAMPLES ON CHAPTER X

1. Find the foci, directrices, eccentricity of the ellipse $4x^2 + 3y^2 = 5$.

2. Find the area of the ellipse $4x^2 + 3y^2 = 5$ (cf. Art. 151, Ex. 9).

3. Show that the polar of a point on a diameter is parallel to the conjugate diameter.

4. Find the equations of the normals at the ends of the latus rectum, and prove that each passes through the end of a minor axis if $e^4 + e^2 = 1$.

5. Show that the four lines from the foci to two points P_1 and P_2 on an ellipse all touch a circle whose center is the pole of P_1P_2.

6. Tangents are drawn from the point (3, 2) to the ellipse

$$x^2 + 4y^2 = 4.$$

Find the equation of the line joining (3, 2) to the middle point of the chord of contact.

7. Find the locus of the center of a circle which passes through the point (0, 3) and touches internally the circle $x^2 + y^2 = 25$.

8. Find the length of the major axis of an ellipse whose minor axis is 10, and whose area is equal to that of a circle whose radius is 8.

9. The minor axis of an ellipse is 6, and the sum of the focal radii for a certain point on the curve is 16; find its major axis, distance between foci, and area.

10. A line of fixed length moves so that its ends remain in the coördinate axes; find the locus generated by any point of the line.

11. Find the locus of the middle points of chords of an ellipse drawn through the positive end of the minor axis.

12. With a given focus and directrix a series of ellipses are drawn; show that the locus of the extremities of their minor axes is a parabola.

13. Show that the line $x \cos \alpha + y \sin \alpha = p$ touches the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

if

$$p^2 = a^2 \cos^2 \alpha + b^2 \sin^2 \alpha.$$

14. Find the locus of the foot of the perpendicular drawn from the center of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to a variable tangent.

15. Prove, analytically, that if the normals to an ellipse pass through its center, the ellipse is a circle.

16. Find the locus of the vertex of a triangle of base $2a$, and such that the product of the tangents of the angles at its base is $\frac{b^2}{c^2}$.

17. The ratio of the subnormals for corresponding points on the ellipse and major auxiliary circle is $\frac{a^2}{b^2}$.

18. Find the equation of the ellipse $9x^2 + 25y^2 = 225$ when referred to its equi-conjugate diameters.

19. Normals at corresponding points on the ellipse, and on the major auxiliary circle, meet on the circle $x^2 + y^2 = (a + b)^2$.

20. Two tangents to an ellipse are perpendicular to each other; find the locus of the middle point of their chord of contact.

21. If P_1 is a point on the director circle, the product of the distances of the center and the pole, respectively, from its polar with respect to the ellipse is constant.

The tangents drawn from the point P to an ellipse make angles θ_1 and θ_2 with the major axis; find the locus of P

22. when $\theta_1 + \theta_2 = 2\alpha$, a constant.

23. when $\tan \theta_1 + \tan \theta_2 = c$, a constant.

Find the locus of the intersection P of two tangents

24. if the sum of the eccentric angles of their points of contact is a constant, equal to 2α.

25. if the difference of the eccentric angles be 120°.

26. Find the locus of the middle points of chords of an ellipse which pass through a given point (h, k).

27. Find the tangents common to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and its mid-circle $x^2 + y^2 = ab$.

CHAPTER XI

The Hyperbola, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

160. Review. The definition of the hyperbola given in Chapter VIII led at once to two standard forms for its equation, viz. (cf. Arts. 116, 118):

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

when the axes of the curve are coincident with the coördinate axes; and

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1,$$

when the axes of the curve are parallel to the coördinate axes, and its center is the point (h, k).

A brief discussion of the first standard form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ showed that the curve has its eccentricity given by the relation $b^2=a^2(e^2-1)$, *i.e.*, by $e^2=\frac{a^2+b^2}{a^2}$; its foci are the two points $(\pm ae, 0)$, and its directrices the lines $x=\pm\frac{a}{e}$ (Art. 116). These results are entirely analogous to the corresponding ones for the ellipse, if it be remembered that $1-e^2$ is positive for the ellipse, while e^2-1 is positive for the hyperbola.

The similarity of the equations of the hyperbola and the ellipse leads to various correspondences in the analytic properties of the curves. For example, the equation

$$\frac{x_1x}{a^2}-\frac{y_1y}{b^2}=1$$

represents the polar of the point (x_1, y_1) with respect to the hyperbola; it represents the chord of contact if the point is outside the hyperbola, and the tangent if the point is upon the curve (Arts. 126, 122). Again, by the method shown in Art. 143, merely replacing b^2 by $-b^2$, it is evident that

$$y = mx \pm \sqrt{a^2m^2 - b^2} \quad . \quad . \quad . \quad [67]$$

is the equation of a tangent to the hyperbola in terms of its slope m. The student will be able in like manner to prove other properties of the hyperbola, analogous to those already shown for the ellipse, using the same methods of derivation.

It was shown, however, in the discussion of Chapter VIII, as also in Art. 48, that the nature of the hyperbola apparently differs widely from that of the ellipse, consisting, as it does, of two open infinite branches instead of one closed oval. It is desired in the present chapter to show some of the most important properties of the hyperbola which correspond to similar properties in the ellipse; and also to prove some special properties which are peculiar to the hyperbola. For the most part, these will be derived for the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$; and the facts summarized above will be assumed.

161. The difference between the focal distances of any point on an hyperbola is constant; it is equal to the transverse axis.

The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ has its foci at the points $F_1 \equiv (-ae, 0)$, $F_2 \equiv (ae, 0)$, with $b^2 = a^2e^2 - a^2$.

Let $P_1 \equiv (x_1, y_1)$ be any given point on the curve, so that

$$y_1^2 = \frac{b^2x_1^2}{a^2} - b^2.$$

Then $\overline{F_1P_1}^2 = (x_1 + ae)^2 + y_1^2 = x_1^2 + 2\,aex_1 + a^2e^2 + y_1^2$

$$= a^2e^2 + 2\,aex_1 + \frac{b^2 + a^2}{a^2}x_1^2 - b^2$$

$$= a^2e^2 + 2\,aex_1 + e^2x_1^2 + a^2 - a^2e^2$$

$$= e^2x_1^2 + 2\,aex_1 + a^2,$$

i.e., $$F_1P_1 = ex_1 + a. \quad . \quad . \quad . \qquad (1)$$

Similarly, $$F_2P_1 = ex_1 - a. \quad . \quad . \quad . \qquad (2)$$

These expressions for the focal distances of a point on the hyperbola are of the same form as those for the ellipse (Art. 144); here, however, $e > 1$.

Subtracting equation (2) from equation (1) gives

$$F_1P_1 - F_2P_1 = 2\,a;$$

hence, *the difference between the focal distances of any point on an hyperbola is constant; it is equal to the transverse axis.*

If the foci are not given, they may be constructed as follows, provided the semi-axes of the curve are known: plot the points $A \equiv (a, 0)$ and $B \equiv (0, b)$; then with the center of the hyperbola as center, and the distance AB as radius, describe a circle; it will cut the transverse axis in the required foci F_1 and F_2, for

$$CF = AB = \sqrt{a^2 + b^2} = \sqrt{a^2e^2} = \pm\, ae.$$

162. Construction of the hyperbola. The property of the preceding article might be taken as a new definition of the hyperbola, viz.: *the hyperbola is the locus of a point the difference of whose distances from two fixed points is constant.* This definition leads at once to the equation of the curve (cf. Ex. 6, p. 67), and also to a method for its construction.

(*a*) *Construction by separate points.* Let $A'A$ be the given difference of the focal distances, — *i.e.*, the transverse axis of the hyperbola, — and F_1 and F_2 the given fixed points, the foci. With either focus, say F_1, as a center, and a radius $A'R > A'A$, describe an arc; then with the other focus as a center, and a radius AR describe an arc cutting the first arcs in the two points P_1. These are points of the hyperbola. Similarly, as many points as desired may be obtained and then connected by a smooth curve, — approximately an hyperbola.

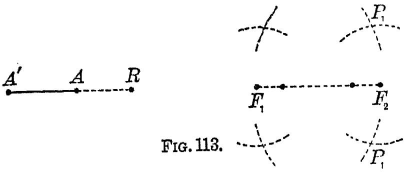


Fig. 113.

(*β*) *Construction by a continuously moving point; the foci being given.* Pivot a straight edge LM at one focus F_1, so that F_1M is greater than the transverse axis $2a$; at M and the other focus F_2 fasten the ends of a string of length l, such that $F_1M = l + 2a$; then a pencil P held against the string and straight edge (see Fig. 114), so as to keep the string always taut, will, while the straight edge revolves about F_1, trace one branch of the hyperbola. By fastening the string at the first focus and the straight edge at the second, the other branch of the curve can be traced.

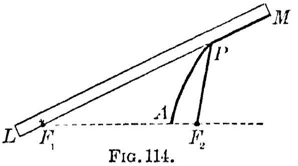


Fig. 114.

163. The tangent and normal bisect internally and externally the angles between the focal radii of the point of contact.

Let F_1 and F_2 be the foci of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, P_1T the tangent, and P_1N the normal at the point $P_1 \equiv (x_1, y_1)$.

Then the equation of P_1T is $\frac{x_1x}{a^2}-\frac{y_1y}{b^2}=1$, and the length of the intercept OT of the tangent is

$$OT=\frac{a^2}{x_1}.$$

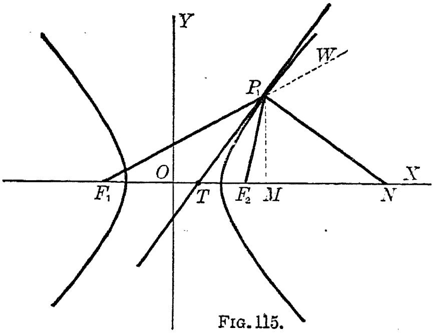


FIG. 115.

Now, in the triangle $F_1P_1F_2$,

$$F_1T=F_1O+OT=ae+\frac{a^2}{x_1}$$

$$=\frac{a}{x_1}(ex_1+a),$$

and
$$TF_2=OF_2-OT=ae-\frac{a^2}{x_1}$$

$$=\frac{a}{x_1}(ex_1-a);$$

but
$$F_1P_1=ex_1+a, \qquad \text{[Art. 161]}$$

and
$$P_1F_2=ex_1-a.$$

Hence
$$F_1T:TF_2=F_1P_1:P_1F_2,$$

and, by plane geometry, the tangent bisects internally the angle between the focal radii. Then, since the normal is perpendicular to the tangent, the normal P_1N bisects the external angle F_2P_1W. These facts suggest a method, analogous to that of Art. 148, for constructing the tangent and normal to an hyperbola at a given point.

164. Conjugate hyperbolas. A curve bearing very close relations to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad . \quad . \quad . \quad (1)$$

is that represented by the equation

$$\frac{y^2}{b^2}-\frac{x^2}{a^2}=1,$$

i.e., by

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1, \quad . \quad . \quad . \quad (2)$$

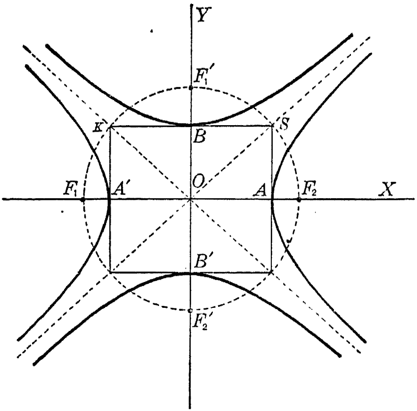


FIG. 116.

in which a and b have the same values as in equation (1). This curve is evidently an hyperbola, and has for its transverse and conjugate axes, respectively, the conjugate and transverse axes of the original, or primary hyperbola. Two such hyperbolas are called **conjugate** hyperbolas; they are sometimes spoken of as the x- and y-hyperbolas, respectively.

It follows at once that the hyperbola (2), conjugate to the hyperbola (1), has for its eccentricity

$$e' = \frac{\sqrt{a^2 + b^2}}{b},$$

for foci the points $(0, \pm be')$, and for directrices the lines

$$y = \pm \frac{b}{e'}.$$

Two conjugate hyperbolas have a common center, and their foci are all at the common distance $\sqrt{a^2 + b^2}$ from this center; *i.e.*, the foci all lie on a circle about the center, having for radius the semi-diagonal OS of the rectangle upon their common axes, and whose sides are tangent to the curves at their vertices. Moreover, when the curves are constructed it will be found that they do not intersect, but are separated by the extended diagonals OS and OK of this circumscribed rectangle, which they approach from opposite sides. These diagonals are examples of a class of lines of great interest in analytic theory, called *asymptotes* (cf. Art. 37, (*c*)).

EXERCISES

1. Construct an hyperbola, given the distance between its foci as 3 cm.

2. Construct an hyperbola, given the distance from directrix to focus as 2 cm.

3. Write the equation of an hyperbola conjugate to the hyperbola $9x^2 - 16y^2 = 144$, and find its axes, foci, and latus rectum. Sketch the figure.

4. Write the equations of the tangent and normal to the hyperbola $16x^2 - 9y^2 = 112$ at the point (4, 4), and find the subtangent and subnormal.

5. Write the equations of the polars of the point (3, 4) with respect to the hyperbola $9x^2 - 16y^2 = 144$ and its conjugate, respectively.

6. For what points of an hyperbola are the subtangent and subnormal equal?

7. Given the hyperbola $9y^2 - 4x^2 = 36$, find the focal radii of the point whose ordinate is $(^-1)$, and abscissa positive.

8. A tangent which is parallel to the line $5x - 4y + 7 = 0$, is drawn to the hyperbola $x^2 - y^2 = 9$; what is the subnormal for the point of contact?

9. What tangent to the hyperbola $\frac{x^2}{10} - \frac{y^2}{12} = 1$ has its y-intercept 2?

10. Find, by equation [67], the two tangents to the hyperbola $4x^2 - 2y^2 = 6$ which are drawn through the point (3, 5).

11. Find the polars of the vertices of an hyperbola with respect to its conjugate hyperbola.

12. Prove that if the crack of a rifle and the thud of the ball on the target are heard at the same instant, the locus of the hearer is an hyperbola.

13. An ellipse and hyperbola have the same axes. Show that the polar of any point on either curve is a tangent to the other.

14. Find the equation of an hyperbola whose vertex bisects the distance from the focus to the center.

15. If e and e' are the eccentricities of an hyperbola and its conjugate, then

$$e^2 + e'^2 = e^2e'^2.$$

16. If e and e' are the eccentricities of two conjugate hyperbolas, then

$$ae = be'.$$

17. Find the eccentricity and latus rectum of the hyperbola

$$y^2 = 4(x^2 + a^2).$$

18. Find the tangents to the hyperbola $9x^2 - 16y^2 = 144$, which, with the tangent at the vertex, form a circumscribed equilateral triangle. Find the area of the triangle.

19. Find the lengths of the tangent, normal, subtangent, and subnormal for the point (3, 2) of the hyperbola $x^2 - 2y^2 = 1$.

165. Asymptotes. If a tangent to an infinite branch of a curve approaches more and more closely to a fixed straight line as a limiting position, when the point of contact moves further and further away on the curve and becomes infinitely distant, then the fixed line is called an **asymptote** of the curve.* More briefly, though less accurately, this definition may be stated as follows: an asymptote to a curve is a tangent whose point of contact is at infinity, but which is not itself entirely at infinity. It is evident that to have an asymptote a curve must have an infinite branch; and this branch may be considered as having two coincident, and infinitely distant, points of intersection with its asymptote. This property will aid in obtaining the equation of the asymptote.

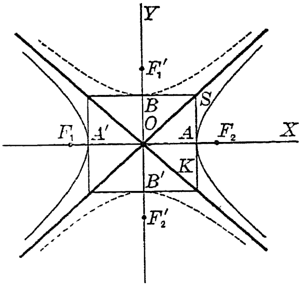


FIG. 117

The hyperbola $$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \quad . \quad . \quad . \quad (1)$$

is cut by the line $$y = mx + c, \quad . \quad . \quad . \quad (2)$$

in two points whose abscissas are given by the equation

$$(a^2m^2-b^2)x^2 + 2\,a^2cmx + a^2b^2 + a^2c^2 = 0. \quad . \quad . \quad (3)$$

If line (2) is an asymptote, the two roots of equation (3) must both become infinite; therefore, by Art. 10,

$$a^2m^2 - b^2 = 0 \quad \text{and} \quad 2\,a^2cm = 0, \quad . \quad . \quad . \quad (4)$$

hence $$c = 0 \quad \text{and} \quad m = \pm\frac{b}{a}.$$

Substituting these values in equation (2), gives

$$y = \frac{b}{a}x, \quad \text{and} \quad y = -\frac{b}{a}x, \quad . \quad . \quad . \quad (5)$$

---

* This definition implies that the distance between a curve and its asymptote becomes infinitely small. McMahon & Snyder, Differential Calculus, Chap. XIV.

and these equations represent the asymptotes of the hyperbola; they are the lines OS and OK in Fig. 117. Therefore, *the hyperbola has two asymptotes, which pass through its center; they are the diagonals of the rectangle described upon its axes.*

Since the equation of the hyperbola conjugate to (1) is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1, \quad . \quad . \quad . \quad (6)$$

and thus differs from equation (1) only in the sign of the second member, which affects only the constant term in equation (3), therefore the equations (4) determine the value of m and c for the asymptotes of the conjugate hyperbola also. It follows that *conjugate hyperbolas have the same asymptotes.*

A second derivation of the equation of the asymptotes of an hyperbola (1) is as follows:

The equation of the tangent to (1) at the point (x_1, y_1) is

$$\frac{x_1x}{a^2}-\frac{y_1y}{b^2}=1, \quad . \quad . \quad . \quad (7)$$

which may be written in the form

$$b^2x = a^2y\frac{y_1}{x_1}+\frac{a^2b^2}{x_1}. \quad . \quad . \quad . \quad . \quad (8)$$

Since (x_1, y_1) is on the curve (1),

therefore $$\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1, \quad i.e., \quad \frac{y_1}{x_1}=\sqrt{\frac{b^2}{a^2}-\frac{b^2}{x_1^2}}. \quad . \quad . \quad (9)$$

Substituting this value of $\frac{y_1}{x_1}$ in equation (8), it becomes

$$b^2x = a^2y\sqrt{\frac{b^2}{a^2}-\frac{b^2}{x_1^2}}+\frac{a^2b^2}{x_1}, \quad . \quad . \quad . \quad (10)$$

which is only another form of the equation of the tangent represented by equations (7) or (8). If now the point of contact (x_1, y_1) moves further and further away, so that $x_1 \doteq \infty$, then the limiting position of the line (10) is represented by $b^2x = a^2y\left(\pm\frac{b}{a}\right)=\pm aby$.

Hence the equations of the asymptotes are: $y=\pm\frac{b}{a}x$ (cf. Art. 156).

The equations of the asymptotes may be combined, by Art. 40, into the one equation which represents both lines, viz.:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0. \quad . \quad . \quad . \qquad [68]$$

166. Relation between conjugate hyperbolas and their asymptotes.

It has been seen that the standard forms for the equations of the primary hyperbola, its asymptotes, and its conjugate hyperbola are, respectively,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \quad . \quad . \quad . \qquad (1)$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0, \quad . \quad . \quad . \qquad (2)$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1. \quad . \quad . \quad . \qquad (3)$$

It will be noticed at once that these three equations differ only in their constant terms; and that the equation of the primary hyperbola (1) differs from that of the asymptotes (2) by the negative of the constant by which the equation of the conjugate hyperbola (3) differs from equation (2). Moreover, this relation between the equations of the three loci must hold when not in their standard forms, *i.e.*, whatever the coördinate axes. For, any transformation of coördinates will affect only the first member of equations (1), (2), and (3), and will affect these in precisely the same way. After the transformation, therefore, the equations of the loci will differ only by a constant (not, however, usually by 1); and the value of the constant in the equation of the asymptotes will be midway between the values of the constants in the equations of the two hyperbolas.

Example 1. An hyperbola having the lines

$$(1)\quad x+2y+3=0 \quad \text{and} \quad (2)\quad 3x+4y+5=0$$

for asymptotes, will have an equation of the form

$$(x+2y+3)(3x+4y+5)+k=0, \quad . \quad . \quad (3)$$

while the equation of its conjugate hyperbola will be

$$(x+2y+3)(3x+4y+5)-k=0. \quad . \quad . \quad (4)$$

If a second condition is imposed upon the hyperbola, *e.g.*, that it shall pass through the point $(1, {}^{-}1)$, then the value of k may be easily found thus: since the curve passes through the point $(1, {}^{-}1)$, therefore by equation (3),

$$(1-2+3)(3-4+5)+k=0\,; \quad \therefore \; k=-8,$$

and the equation of the hyperbola is

$$(x+2y+3)(3x+4y+5)-8=0,$$

that is, $$3x^2+10xy+8y^2+14x+22y+7=0\,; \quad . \quad (5)$$

and the equation of the conjugate hyperbola is

$$3x^2+10xy+8y^2+14x+22y+23=0.$$

Example 2. The equation of the asymptotes of the hyperbola

$$3x^2-14xy-5y^2+7x+13y-8=0 \quad . \quad . \quad . \quad (1)$$

differs from equation (1) by a constant only, hence it is of the form

$$3x^2-14xy-5y^2+7x+13y+k=0. \quad . \quad . \quad (2)$$

Now equation (2) represents a pair of straight lines, therefore its first member can be factored, and, by Art. 67, [17]

$$-15k-\tfrac{1274}{4}-\tfrac{507}{4}+\tfrac{245}{4}-49k=0\,;$$

i.e., $$64k=-384, \quad \text{whence} \quad k=-6.$$

Therefore the equation of the asymptotes is

$$3x^2-14xy-5y^2+7x+13y-6=0,$$

i.e., $$(3x+y-2)(x-5y+3)=0\,;$$

and the equation of the conjugate hyperbola is

$$3x^2 - 14xy - 5y^2 + 7x + 13y - 4 = 0.$$

167. Equilateral or rectangular hyperbola. If the axes of an hyperbola are equal, so that $a = b$, its equation has the form

$$x^2 - y^2 = a^2, \qquad . \quad . \quad . \qquad (1)$$

and its eccentricity $e = \sqrt{2}$. Its conjugate hyperbola has the equation

$$x^2 - y^2 = -a^2; \qquad . \quad . \quad . \qquad (2)$$

with the same eccentricity and the same shape; while its asymptotes have the equations

$$x = \pm y, \qquad . \quad . \quad . \qquad (3)$$

and are therefore the bisectors of the angles formed by the axes of the curves; hence the asymptotes of these hyperbolas are perpendicular to each other. The hyperbola whose axes are equal is therefore called an **equilateral,** or a **rectangular hyperbola,** according as it is thought of as having equal axes or asymptotes at right angles.

EXERCISE

1. Find the asymptotes of the hyperbola $9x^2 - 16y^2 = 25$, and the angle between them.

2. What are the poles of the asymptotes of the hyperbola

$$9x^2 - 16y^2 = 25$$

with reference to the curve?

3. If the vertex lies two thirds of the distance from the center to the focus, find the equations of the hyperbola, and of its asymptotes.

4. If a line $y = mx + c$ meets the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in one finite and one infinitely distant point, the line is parallel to an asymptote.

5. Show that, in an equilateral hyperbola, the distance of a point from the center is a mean proportional between its focal distances.

6. Find the equation of the hyperbola passing through the point (0, 7), and having for asymptotes the lines

$$2x - y = 7, \text{ and } 3x + 3y = 5 \text{ (cf. Art. 166).}$$

7. Write the equation of the hyperbola conjugate to that of Ex. 6.

8. Find the equations of the asymptotes of the hyperbola

$$2x^2 - xy - 2x = y^2 + y + 6;$$

also find the equation of the conjugate hyperbola.

9. Find the equation of the asymptotes of the hyperbola

$$3x^2 + 34xy + 11y^2 - x + 21y = 0.$$

10. Find the equation of the hyperbola conjugate to

$$9x^2 - 16y^2 + 36x + 160y = 508.$$

11. Prove that a perpendicular from the focus to an asymptote of an hyperbola is equal to the semi-conjugate axis.

12. The asymptotes meet the directrices of the x-hyperbola on the x-auxiliary circle, and of the conjugate hyperbola on the y-auxiliary circle.

13. The circle described about a focus, with a radius equal to half the conjugate axis, will pass through the intersections of the asymptotes and a directrix.

14. The line from the center C to the focus F of an hyperbola is the diameter of a circle that cuts an asymptote at P; show that the chords CP and FP are equal, respectively, to the semi-transverse and semi-conjugate axes.

168. The hyperbola referred to its asymptotes. If the asymptotes of an hyperbola are chosen as the coördinate axes, their equations will be $x = 0$ and $y = 0$, respectively; or, combined in one equation,

$$xy = 0. \quad . \quad . \quad . \quad (1)$$

By the reasoning of Art. 166, it follows that the equation of the hyperbola, — which differs from that of its asymptotes by a constant, — is

$$xy = k, \quad . \quad . \quad . \quad (2)$$

wherein the value of the constant k is to be determined by an additional assigned condition concerning the curve; *e.g.*, that it shall pass through a given point.

The value of this constant, in terms of a and b, can in general be found most easily by making the proper transformation of coördinates upon the equation of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1. \quad . \quad . \quad . \qquad (3)$$

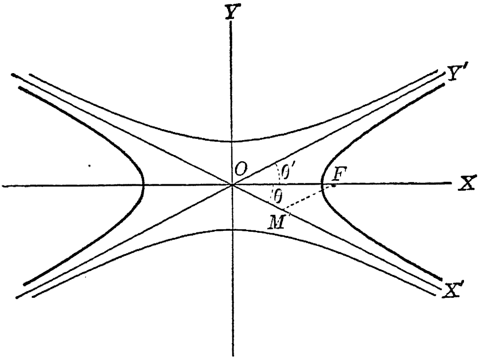


FIG. 118.

The new x-axis makes the angle θ, the new y-axis the angle θ', with the old x-axis, such that

$$\tan\theta=-\frac{b}{a},\quad \tan\theta'=\frac{b}{a}.$$

Hence
$$\sin\theta=-\sin\theta'=\frac{-b}{\sqrt{a^2+b^2}},$$

and
$$\cos\theta=+\cos\theta'=\frac{a}{\sqrt{a^2+b^2}};$$

therefore the formulas [24] for transformation,

$$x=x'\cos\theta+y'\cos\theta',\quad y=x'\sin\theta+y'\sin\theta',$$

become in this case

$$x=\frac{a}{\sqrt{a^2+b^2}}(x'+y'),\quad y=\frac{b}{\sqrt{a^2+b^2}}(x'-y'). \quad . \qquad (4)$$

Applying this transformation, equation (3) becomes

$$\frac{x'^2 + 2\,x'y' + y'^2}{a^2 + b^2} - \frac{x'^2 - 2\,x'y' + y'^2}{a^2 + b^2} = 1\,;$$

that is, dropping the accents,

$$xy = \frac{a^2 + b^2}{4}, \quad . \quad . \quad . \qquad [69]$$

which is the desired equation of the hyperbola when referred to its asymptotes as coördinate axes.

The equation of the conjugate hyperbola is then

$$xy = -\frac{a^2 + b^2}{4}. \quad . \quad . \quad . \qquad (5)$$

Remembering the relation $b^2 = a^2(e^2 - 1)$, it will be seen that the value of the constant term in equation (2) may be written

$$k = \frac{a^2 + b^2}{4} = \frac{a^2e^2}{4} = c^2,$$

so that c is half the distance of the focus from the center of the curve. Again, the coördinates of the foci, $x = \pm ae$, $y = 0$, become after the transformation (4),

$$x = y = \pm\frac{a^2 + b^2}{2\,a}; \quad . \quad . \quad . \qquad (6)$$

and the equations of the directrices, $x = \pm\frac{a}{e}$, become

$$x + y = \pm\, a. \quad . \quad . \quad . \qquad (7)$$

169. The tangent to the hyperbola $xy = c^2$. The equation of the tangent to the hyperbola

$$xy = c^2, \quad . \quad . \quad . \qquad (1)$$

at any given point (x_1, y_1), may be easily derived by the secant method (cf. Arts. 84, 122). Let $P_1 \equiv (x_1, y_1)$ and $P_2 \equiv (x_2, y_2)$ be two points on the curve; then

$$x_1y_1 = c^2, \quad . \quad . \quad (2) \quad \text{and} \quad x_2y_2 = c^2. \quad . \quad . \quad (3)$$

The equation of the line through P_1 and P_2 is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1),$$

wherein $\frac{y_2 - y_1}{x_2 - x_1}$ must have a value determined by equations (2) and (3), hence

$$m = \frac{\frac{c^2}{x_2} - \frac{c^2}{x_1}}{x_2 - x_1} = \frac{c^2}{x_1 x_2} \cdot \frac{x_1 - x_2}{x_2 - x_1} = -\frac{c^2}{x_1 x_2}.$$

The equation of the secant line P_1P_2 is therefore

$$y - y_1 = -\frac{c^2}{x_1 x_2}(x - x_1). \quad . \quad . \quad . \quad (4)$$

If now the point P_2 becomes coincident with P_1, equation (4) becomes

$$y - y_1 = -\frac{c^2}{x_1^2}(x - x_1),$$

which may be reduced by equation (2) to

$$\frac{x}{x_1} + \frac{y}{y_1} = 2, \quad . \quad . \quad . \quad [70]$$

or to

$$y_1 x + x_1 y = 2\,c^2,$$

which is the required equation of the tangent at the point $P_1 \equiv (x_1, y_1)$ of the curve.

170. Geometric properties of the hyperbola. Equation [69] states the following intrinsic property for the hyperbola, $P_1 \equiv (x_1, y_1)$ being any point on the curve (Fig. 119).

$$4\,MP_1 \cdot LP_1 = \overline{OF}^2;$$

that is, *for every point of the hyperbola, four times the product of its distances from the asymptotes, measured parallel to the asymptotes respectively, is equal to the square of the distance from the center to the focus; and is therefore constant.*

Again, 2θ being the angle between the asymptotes, equation [69] may be written

$$xy \sin 2\theta = \frac{\sqrt{a^2+b^2}}{2} \frac{\sqrt{a^2+b^2}}{2} \sin 2\theta. \quad . \quad . \quad (1)$$

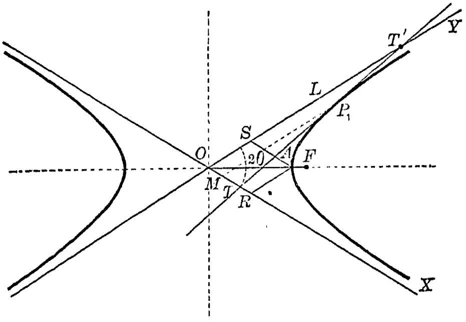


FIG. 119.

Now $xy \sin 2\theta$ is the area of the parallelogram OMP_1L, constructed upon the coördinates of the point P_1 of the hyperbola; and since the coördinates of the vertex A are $x = y = \frac{\sqrt{a^2+b^2}}{2}$, the second member of equation (1) is the area of the rhombus $ORAS$, constructed upon the coördinates of the vertex. Therefore, *the area of the parallelogram formed by the asymptotes and lines parallel to them drawn from any point of an hyperbola, is constant; it is equal to the rhombus similarly drawn from the vertex of the curve.*

The equation of the tangent to the hyperbola

$$xy = c^2, \quad . \quad . \quad . \quad (2)$$

at the point P, is

$$\frac{x}{x_1} + \frac{y}{y_1} = 2. \quad . \quad . \quad . \quad (3)$$

The x-intercept of this tangent is $OT = 2x_1$; hence if OT' be the y-intercept, and M the foot of the ordinate of P_1, then from the similar triangles MTP_1 and OTT',

$$TP_1 : TT' = MT : OT = x_1 : 2x_1 = 1 : 2.$$

Hence, *the segment of any tangent to an hyperbola between the asymptotes is bisected by the point of contact.*

The tangent (3) has the intercepts on the x-axis and y-axis, respectively,

$$OT = 2x_1, \quad OT' = 2y_1.$$

Then $$OT \cdot OT' = 4x_1y_1. \quad . \quad . \quad . \qquad (4)$$

But since (x_1, y_1) is a point of the hyperbola

$$4x_1y_1 = a^2 + b^2,$$

hence $$OT \cdot OT = a^2 + b^2, \quad . \quad . \quad . \qquad (5)$$

i.e., *the rectangle formed by the intercepts which any tangent to the hyperbola makes upon the asymptotes is constant; it is equal to the sum of the squares upon the semi-axes.*

Moreover, equation (5) may be written

$$OT \cdot OT' \frac{\sin 2\theta}{2} = \frac{a^2 + b^2}{2} \sin 2\theta; \quad . \quad . \quad . \qquad (6)$$

but $$\sin 2\theta = 2 \sin\theta \cos\theta = 2 \frac{b}{\sqrt{a^2 + b^2}} \frac{a}{\sqrt{a^2 + b^2}} = \frac{2ab}{a^2 + b^2},$$

hence (6) bcomes $$\frac{OT \cdot OT'}{2} \sin 2\theta = ab; \quad . \quad . \quad . \quad . \qquad (4)$$

that is, *the triangle formed by any tangent to an hyperbola and its asymptotes is constant; it is equal to the rectangle upon the semi-axes.*

EXERCISES

1. Find the equation of the hyperbola $9x^2 - 16y^2 = 25$ when referred to its asymptotes as axes.

2. Find the semi-axes, eccentricity, and the vertices, of the hyperbola $xy = 4$, the angle between the axes (asymptotes) being 90°.

3. Find the semi-axes, eccentricity, vertices, and the foci, of the hyperbola $xy = -12$, the angle between the axes being 60°.

4. Prove that the segments of any line which are intercepted between an hyperbola and its asymptotes are equal.

5. Express the angle between the asymptotes of an hyperbola in terms of e; *i.e.*, in terms of the eccentricity of the hyperbola.

6. The segment of a tangent to an hyperbola intercepted by the conjugate hyperbola is bisected at the point of contact.

7. Show that the pole of any tangent to the rectangular hyperbola $xy = c^2$, with respect to the circle $x^2 + y^2 = a^2$, lies on a concentric and similarly placed rectangular hyperbola.

8. Prove that the asymptotes of the hyperbola $xy = hx + ky$ are $x = k$, and $y = h$.

9. Derive the equation of the tangent to the curve $xy = hx + ky$ at the point $P \equiv (x_1, y_1)$ on the curve.

171. Diameters. A diameter has already been defined (Art. 129) as the locus of the middle points of a system of parallel chords, and in Art. 152 the equation was derived for a diameter of an ellipse. By the same method, if a system of parallel chords of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

have the common slope m, the equation of the corresponding diameter will be found to be

$$y = \frac{b^2}{a^2 m} x. \quad . \quad . \quad . \quad [71]$$

This equation shows that *every diameter of the hyperbola passes through the center.*

Conversely, it is true, as in the case of the ellipse, that every chord of the hyperbola through the center is a diameter. That chord of the original set which passes through the center is the diameter *conjugate* to [71]; and its equation is

$$y = mx. \quad . \quad . \quad . \quad [72]$$

Letting m' be the slope of a diameter, and m that of its conjugate, the essential condition that two diameters should be conjugate to each other is that (cf. Art. 153)

$$mm' = \frac{b^2}{a^2}. \quad . \quad . \quad . \qquad [73]$$

172. Properties of conjugate diameters of the hyperbola. (α) It is clear that the condition

$$mm' = \frac{b^2}{a^2} \quad . \quad . \quad . \qquad [73]$$

holds also for the hyperbola

$$\frac{x^2}{-a^2} + \frac{y^2}{b^2} = 1,$$

which is conjugate to the given hyperbola; for, replacing a^2 by $-a^2$ and $-b^2$ by b^2 leaves equation [73] unchanged. Hence, *diameters which are conjugate to each other for a given hyperbola are conjugates also for the conjugate of that hyperbola.*

(β) The axes of the hyperbola are clearly diameters of the curve. They are perpendicular to each other, and therefore satisfy the relation

$$mm' = -1.$$

Comparing this condition with that of equation [73], it follows that *the transverse and conjugate axes of the hyperbola are the only pair of perpendicular conjugate diameters* (cf. (β) p. 255).

If $a = b$, the condition [73] reduces to

$$mm' = 1;$$

therefore (Art. 16), in the rectangular hyperbola the sum of the angles which a pair of conjugate diameters make with the transverse axis is 90° (cf. Art. 156).

(γ) Since in equation [73] the product mm' is positive, it follows that the angles which conjugate diameters make with the transverse axis are both acute, or both obtuse. Moreover,

$$\text{if } m < \pm\frac{b}{a}, \text{ then } m' > \pm\frac{b}{a};$$

and the diameters lie on opposite sides of an asymptote. *Two conjugate diameters lie in the same quadrant formed by the axes of the hyperbola, on opposite sides of the asymptote* (cf. Art. 155 (a)).

(δ) An asymptote passes through the center of an hyperbola, hence may be regarded as a diameter. Its slope is

$$m = \pm\frac{b}{a}, \qquad \therefore\ m' = \pm\frac{b}{a};$$

hence, *an asymptote regarded as a diameter is its own conjugate;* it may be called a *self-conjugate* diameter.

This is a limiting case of (γ) above.

(ϵ) It follows from this last fact that if a diameter intersects a given hyperbola, then the conjugate diameter does not intersect it, but cuts the conjugate hyperbola. It is customary and useful to define as the extremities of the conjugate diameter its points of intersection with the conjugate hyperbola. With this limitation, it follows from (a) of this article, that, as in the ellipse, *each of two conjugate diameters bisects the chords parallel to the other.*

(ζ) As a limiting case of this last proposition, also, it is evident that *the tangent at the end of a diameter is parallel to the conjugate diameter.*

By reasoning entirely analogous to that given in Art. 155, for the ellipse, properties similar to those there given may be derived for the hyperbola. They are included in the following exercises, to be worked out by the student.

EXERCISES

1. Find the equation of the diameter of the hyperbola

$$9x^2 - 16y^2 = 25$$

which bisects the chords $y = 3x + b$.

Find also the conjugate diameter.

2. Find, for the hyperbola of Ex. 1, a diameter through the point (1, 1), and its conjugate.

3. Find the diameter of the hyperbola $\frac{x^2}{16} - \frac{y^2}{25} = 1$ which is conjugate to the diameter $x - 3y = 0$.

4. Find the equation of a chord of the hyperbola $12x^2 - 9y^2 = 108$, which is bisected at the point (4, 2).

5. Lines from any point of an equilateral hyperbola to the extremities of a diameter make equal angles with the asymptotes.

6. Show that, in an equilateral hyperbola, conjugate diameters make equal angles with the asymptotes.

7. The difference of the squares of two conjugate semi-diameters is constant; it is equal to the difference of the squares of the semi-axes.

8. The angle between two conjugate diameters is $\sin^{-1}\frac{ab}{a'b'}$.

9. The polar of one end of a diameter of an hyperbola, with reference to the conjugate hyperbola, is the tangent at the other end of the given diameter.

10. Tangents at the ends of a pair of conjugate diameters intersect on an asymptote.

173. Supplemental chords. As previously defined, chords of a curve are supplemental when drawn from any point of the curve to the extremities of a diameter. If, in the analytic work of Art. 157, b^2 is replaced by $-b^2$, then, if m and m' are the slopes of two supplemental chords of the hyperbola, they must satisfy the relation

$$mm' = \frac{b^2}{a^2}. \quad . \quad . \quad . \qquad (1)$$

But this is (see Eq. [73]) the condition that exists between the slopes of two conjugate diameters. Therefore, *supplemental chords are parallel to a pair of conjugate diameters.*

For the equilateral hyperbola, *i.e.*, when $a = b$, this relation has the special value

$$mm' = 1, \quad . \quad . \quad . \qquad (2)$$

and, therefore, the sum of the acute angles which a pair of supplementary chords of the equilateral hyperbola make with its transverse axis is 90° (cf. Art. 172 (β)).

174. Equations representing an hyperbola, but involving only one variable.

(a) *Eccentric angle.* In the theory of the hyperbola, the auxiliary circles described upon the axes of the curve as diameters are not as useful as the corresponding circles for the ellipse, since the ordinate for a point on the hyperbola does not cut the x-auxiliary circle, and, therefore, there is no simple construction for the eccentric angle. It is, however, sometimes desirable to express by means of a single variable the condition that a point shall be on an hyperbola; and for this purpose the equations

$$x = a \sec \phi, \; y = b \tan \phi, \qquad [74]$$

similar to equations [60], may be used; for these evidently satisfy the equation of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

since
$$\sec^2 \phi - \tan^2 \phi = 1.$$

The angle ϕ may be defined as the **eccentric angle** for the hyperbola, and the corresponding point of the curve may be constructed as follows:

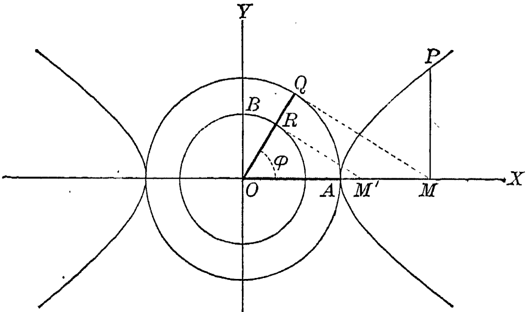


FIG. 120.

Draw the auxiliary circles, and any $\angle AOQ = \phi$. At the points R and Q, where the terminal side of ϕ cuts the circles, draw tangents cutting the transverse axis in the points M' and M, respectively. Erect at M an ordinate MP equal to RM'; its extremity P is a point of the hyperbola. For, in the right triangle OMQ,

$$OM\cos\phi = OQ, \quad i.e., \; OM = a\sec\phi;$$

and, in the right triangle $OM'R$,

$$M'R = OR\tan\phi, \quad i.e., \; M'R = b\tan\phi.$$

But for the point P,

$$x = OM, \; y = MP = M'R;$$

hence

$$x = a\sec\phi, \; y = b\tan\phi,$$

and P is a point on the hyperbola.*

The eccentric angle for any given point, P, of an hyperbola is easily obtained. Draw the ordinate MP, and from its foot, M, draw a tangent MQ to the x-auxiliary circle; then the angle MOQ is the eccentric angle corresponding to P.

(β) The equation of the hyperbola referred to its asymptotes, viz. $xy = c^2$, is satisfied by the coördinates $x = ct$, $y = \frac{c}{t}$, whatever the values of t. The use of this single independent variable t is sometimes convenient in dealing with points on the hyperbola.*

EXAMPLES ON CHAPTER XI

1. Write the equation of an hyperbola whose transverse axis is 8, and the conjugate axis one half the distance between the foci.

2. Find the equation of that diameter of the hyperbola $16x^2 - 9y^2 = 144$ which passes through the point $(5, \frac{16}{3})$; also find the coördinates of the extremities of the conjugate diameter.

3. Assume the equation of the hyperbola, and show that the difference of the focal distances is constant.

4. Find the locus of the vertex of a triangle of given base $2c$, if the difference of the two other sides is a constant, and equal to $2a$.

5. Find the locus of the vertex of a triangle of given base, if the difference of the tangents of the base angles is constant.

6. Find an expression for the angle between any pair of conjugate diameters of an hyperbola.

7. Show that two concentric rectangular hyperbolas, whose axes meet at an angle of 45°, cut each other orthogonally.

---

* The forms of this article are useful in the differential calculus.

8. The portions of any chord of an hyperbola intercepted between the curve and its conjugate are equal.

SUGGESTION. Draw a tangent parallel to the line in question.

9. The coördinates of a point are $a \tan(\theta + \alpha)$ and $b \tan(\theta + \beta)$; prove that the locus of the point, as θ varies, is an hyperbola.

10. Prove that the asymptotes of the hyperbola $xy = 3x + 5y$ are $x = 5$ and $y = 3$.

11. If the coördinate axes are inclined at an angle ω, find the equation of an hyperbola whose asymptotes are the lines $x = 2$ and $y = -3$, respectively, and which passes through the point $(2, 1)$.

12. Find the coördinates of the points of contact of the common tangents to the hyperbolas,

$$x^2 - y^2 = 3a^2, \text{ and } xy = 2a^2.$$

13. If a right-angled triangle be inscribed in a rectangular hyperbola, prove that the tangent at the right angle is perpendicular to the hypothenuse.

14. Show that the line $y = mx + 2k\sqrt{-m}$ always touches the hyperbola $xy = k^2$; and that its point of contact is $\left(\frac{c}{\sqrt{-m}}, c\sqrt{-m}\right)$.

15. Find the point of the hyperbola $xy = 12$ for which the subtangent is 4. Find the subnormal for the same point.

16. Find the polar of the point $(5, 3)$ on the hyperbola $x^2 - 2y^2 = 7$, with respect to the conjugate hyperbola. Show that this line is tangent to the given hyperbola, at the other end of the diameter from $(5, 3)$.

17. If an ellipse and hyperbola have the same foci, they intersect at right angles.

18. Find tangents to the hyperbola $2y^2 - 16x^2 = 1$ which are perpendicular to its asymptotes.

19. Find normals to the hyperbola $\frac{(x-3)^2}{16} - \frac{(y-2)^2}{9} = 1$ which are parallel to its asymptotes. Find the polar of their point of intersection.

20. Show that, in an equilateral hyperbola, conjugate diameters are equally inclined to the asymptotes.

21. Show that two conjugate diameters of a rectangular hyperbola are equal.

22. Show that, in an equilateral hyperbola, two diameters at right angles to each other are equal. Show also that this follows from Ex. 21.

23. Find the sum of two focal chords which are, respectively, parallel to two conjugate diameters.

24. Find the common tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and its mid-circle $x^2+y^2=ab$.

25. In the hyperbola $25x^2-16y^2=400$, find the conjugate diameters that cut each other at an angle of 45°.

26. The latus rectum of an hyperbola is a third proportional to the two axes.

27. The polars of any point (h, k) with respect to conjugate hyperbolas are parallel.

28. The sum of the eccentric angles of the extremities of two conjugate diameters of an hyperbola is equal to 90°; *i.e.*, $\phi+\phi'=90°$.

29. Find the equation of a line through the focus of an hyperbola and the focus of its conjugate, and find the pole of that line.

30. Find the asymptotes of the hyperbola $xy-3x-2y=0$. What is the equation of the conjugate hyperbola?

31. Show that the y-axis is an asymptote of the hyperbola

$$2xy+3x^2+4x=9.$$

What is the equation of the other asymptote? Of the conjugate hyperbola?

32. If two tangents are drawn from an external point to an hyperbola, they will touch the same or opposite branches of the curve according as the given point lies between or outside of the asymptotes.

CHAPTER XII

GENERAL EQUATION OF THE SECOND DEGREE

$$Ax^2 + 2\,Hxy + By^2 + 2\,Gx + 2\,Fy + C = 0$$

175. General equation of the second degree in two variables. Thus far only special equations of the second degree have been studied; they have all been of the form

$$Ax^2 + By^2 + 2\,Gx + 2\,Fy + C = 0, \quad . \quad . \quad . \quad (1)$$

i.e., they have been free from the term containing the product of the variables. In Arts. 107, 113, and 119 it is shown that equation (1) represents a conic section having its axes parallel to the coördinate axes. It still remains to be shown, however, that the most general equation of the second degree, viz.

$$Ax^2 + 2\,Hxy + By^2 + 2\,Gx + 2\,Fy + C = 0, \quad . \quad . \quad . \quad (2)$$

also represents a conic section. To prove this it is only necessary to show that, by a suitable change of the coördinate axes, equation (2) may be reduced to the form of equation (1).

If equation (2) be referred to new axes, OX' and OY', say, making an angle θ with the corresponding given axes; and if the new coördinates of any point on the curve be x' and y', the old coördinates of the same point being x and y; then (Art. 72)

$$x = x' \cos\theta - y' \sin\theta, \text{ and } y = x' \sin\theta + y' \cos\theta. \quad . \quad . \quad (3)$$

Substituting these values (3) in equation (2), it becomes

$$A(x'\cos\theta-y'\sin\theta)^2+2H(x'\cos\theta-y'\sin\theta)(x'\sin\theta+y'\cos\theta)$$
$$+B(x'\sin\theta+y'\cos\theta)^2+2G(x'\cos\theta-y'\sin\theta)$$
$$+2F(x'\sin\theta+y'\cos\theta)+C=0, \quad . \quad . \quad . \qquad (4)$$

which, being expanded and re-arranged, becomes:

$$x'^2(A\cos^2\theta+2H\sin\theta\cos\theta+B\sin^2\theta)$$
$$+x'y'(-2A\sin\theta\cos\theta-2H\sin^2\theta+2H\cos^2\theta+2B\sin\theta\cos\theta)$$
$$+y'^2(A\sin^2\theta-2H\sin\theta\cos\theta+B\cos^2\theta)$$
$$+x'(2G\cos\theta+2F\sin\theta)$$
$$+y'(-2G\sin\theta+2F\cos\theta)+C=0. \quad . \quad . \quad . \qquad (5)$$

This transformed equation (5) will be free from the term containing the product $x'y'$ if θ be so chosen that

$$-2A\sin\theta\cos\theta-2H\sin^2\theta+2H\cos^2\theta+2B\sin\theta\cos\theta=0,$$

i.e., if $\quad 2H(\cos^2\theta-\sin^2\theta)=(A-B)2\sin\theta\cos\theta,$

i.e., if $\quad 2H\cdot\cos 2\theta=(A-B)\sin 2\theta,$

or finally, if $\quad \tan 2\theta=\dfrac{2H}{A-B}. \quad . \quad . \quad . \qquad (6)$

Moreover, it is always possible to choose a positive acute angle θ so as to satisfy this last equation whatever may be the numbers represented by A, B, and H.

Having chosen θ so as to satisfy equation (6), and having substituted the values of $\sin\theta$ and $\cos\theta$ in equation (5), that equation reduces to

$$A'x'^2+B'y'^2+2G'x'+2F'y'+C=0, \quad . \quad . \qquad (7)$$

(wherein A', B', $\cdots$ represent the new coefficients)

and therefore represents a conic section with its axes parallel to the new coördinate axes. But equation (7) represents the same locus as equation (2); hence it is proved that, in rectangular coördinates, *every equation of the form*

$$Ax^2 + 2Hxy + By^2 + Gx + 2Fy + C = 0$$

represents a conic section whose axes are inclined at an angle θ to the given coördinate axes, where θ is determined by the equation

$$\tan 2\theta = \frac{2H}{A - B}.$$

It is to be noted that the constant term C has remained unchanged by the transformation given above.

The next article will illustrate the application of this method to numerical equations. It is to be observed that this method is entirely general, and enables one to fully determine the conic represented by any given numerical equation of the second degree.

NOTE. In the proof just given that every equation of the second degree represents a conic section, it is assumed that the given axes are at right angles. This restriction may, however, be removed; for if they are not at right angles, a transformation may be made to rectangular axes having the same origin (cf. Arts. 74, 75), and the equation will have its form and degree left unchanged; after which the proof already given applies.

176. **Illustrative examples.** EXAMPLE 1. Given the equation

$$-x^2 + 4xy - y^2 - 4\sqrt{2}x + 2\sqrt{2}y - 11 = 0, \quad . \quad . \quad . \quad (1)$$

to determine the nature and position of its locus.

Turn the axes through an angle θ, *i.e.*, substitute for x and y, respectively, $x'\cos\theta - y'\sin\theta$ and $x'\sin\theta + y'\cos\theta$; equation (1) then becomes

$$\begin{aligned} x'^2(&-\cos^2\theta + 4\sin\theta\cos\theta - \sin^2\theta) \\ &+ x'y'(+2\sin\theta\cos\theta + 4\cos^2\theta - 4\sin^2\theta - 2\sin\theta\cos\theta) \\ &- y'^2(\sin^2\theta + 4\sin\theta\cos\theta + \cos^2\theta) \\ &- x'(4\sqrt{2}\cos\theta - 2\sqrt{2}\sin\theta) \\ &+ y'(+4\sqrt{2}\sin\theta + 2\sqrt{2}\cos\theta) - 11 = 0. \quad . \quad . \quad . \quad (2) \end{aligned}$$

The coefficient of $x'y'$ in equation (2) reduces to $4(\sin^2\theta - \cos^2\theta)$; it will therefore be zero if $\sin\theta = \cos\theta$, *i.e.*, if $\theta = 45°$.*

If $\theta = 45°$, then $\sin\theta = \cos\theta = \dfrac{1}{\sqrt{2}}$, and this value of $\sin\theta$ and $\cos\theta$ being substituted in equation (2), it becomes

$$x'^2 - 3y'^2 - 2x' + 6y' - 11 = 0, \qquad . \quad . \quad . \quad (3)$$

which represents the same locus as is represented by equation (1); the difference in the form of the two equations being due to the fact that the axes to which equation (3) is referred make an angle of 45° with the axes to which equation (1) is referred.

Equation (3) may be written in the form

$$(x'-1)^2 - 3(y'-1)^2 = 9,$$

i.e.,

$$\frac{(x'-1)^2}{3^2} - \frac{(y'-1)^2}{(\sqrt{3})^2} = 1, \qquad . \quad . \quad . \quad (4)$$

which represents an hyperbola (cf. Art. 118). Its center is at the point (1, 1); the transverse axis is parallel to the x'-axis; the semi-axes are of length 3 and $\sqrt{3}$, respectively; the eccentricity is $e = \frac{2}{3}\sqrt{3}$; the foci are at the points $F \equiv (1 + 2\sqrt{3}, 1)$ and $F' \equiv (1 - 2\sqrt{3}, 1)$, respectively; the directrices have the equations

$$x' = 1 + \tfrac{3}{2}\sqrt{3}$$

and $\quad x' = 1 - \tfrac{3}{2}\sqrt{3},$

respectively; and the latus rectum is 2. All these results refer to the new axes, of course, and the locus is that represented in Fig. 121.

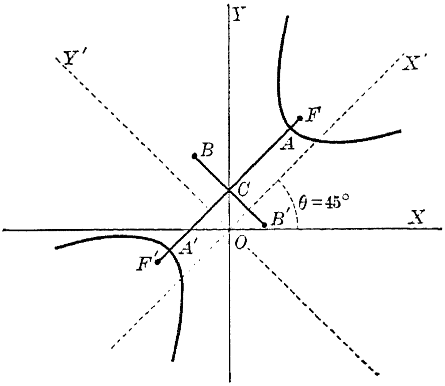


FIG. 121.

---

* This accords with a result of the preceding article, viz. that to free an equation from its xy-term it is only necessary to turn the axes through a positive acute angle determined by $\tan 2\theta = \dfrac{2H}{A-B}$. In the present problem $H = +2$ and $A = B = -1$, hence $\tan 2\theta = \infty$ and $\theta = 45°$.

EXAMPLE 2. Given the equation

$$4x^2 + 4xy + y^2 - 18x + 26y + 64 = 0, \quad . \quad . \quad . \quad (5)$$

to determine the nature and position of its locus. Turn the axes through an angle θ, *i.e.*, substitute for x and y, respectively, $x' \cos\theta - y' \sin\theta$ and $x' \sin\theta + y' \cos\theta$; equation (5) then becomes

$$\begin{aligned} x'^2(4\cos^2\theta &+ \sin^2\theta + 4\sin\theta\cos\theta) \\ &+ x'y'(-8\cos\theta\sin\theta + 2\cos\theta\sin\theta - 4\sin^2\theta + 4\cos^2\theta) \\ &+ y'^2(4\sin^2\theta + \cos^2\theta - 4\sin\theta\cos\theta) \\ &+ x'(-18\cos\theta + 26\sin\theta) \\ &+ y'(18\sin\theta + 26\cos\theta) + 64 = 0, \quad . \quad . \quad . \quad (6) \end{aligned}$$

in which θ is to be so determined that the coefficient of $x'y'$ shall be zero. On placing this coefficient equal to zero, it is at once seen that $\tan 2\theta = \frac{4}{3}$, from which it follows (cf. exercise 3, Art. 16, second method) that

$$\sin 2\theta = \tfrac{4}{5} \text{ and } \cos 2\theta = \tfrac{3}{5};$$

remembering that $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$, it is easily deduced that $\sin\theta = \frac{1}{\sqrt{5}}$ and $\cos\theta = \frac{2}{\sqrt{5}}$.

Substituting these values in equation (6), it becomes

$$5x'^2 - 2\sqrt{5}\,x' + 14\sqrt{5}\,y' + 64 = 0,$$

i.e.,

$$\left(x' - \frac{1}{\sqrt{5}}\right)^2 = -\frac{14}{\sqrt{5}}\left(y' + \frac{63}{14\sqrt{5}}\right); \quad . \quad . \quad (7)$$

which is the equation of a parabola whose vertex is at the point

$$\left(\frac{1}{\sqrt{5}}, -\frac{63}{14\sqrt{5}}\right);$$

whose focus is at the point $\left(\frac{1}{\sqrt{5}}, -\frac{8}{\sqrt{5}}\right)$, whose axis coincides with the negative end of the y'-axis, and whose latus rectum is $\frac{14}{\sqrt{5}}$. All these results refer to the new axes; the locus of the above equation is given in Fig. 79, p. 178 (Art. 108).

EXERCISES

1. For the hyperbola in Fig. 121 find the coördinates of the center and of the foci, and also the equations of its axes and directrices, all referred to the axes OX and OY.

By first removing the xy-term, determine the nature and position of the loci represented by the following equations. Also plot the curves.

2. $y^2 - 2\sqrt{3}\,xy + 3x^2 + 6x - 4y + 5 = 0.$

3. $x^2 - 4xy + 3y^2 - x - y = 0.$

4. $3x^2 + 2xy + 3y^2 - 16y + 23 = 0.$

5. $x^2 - 2xy + y^2 - 6x - 6y + 9 = 0.$

177. Test for the species of a conic. It is often desirable to know the species of a conic represented by a given equation even when it may not be necessary to determine fully the position of the curve. Remembering that every equation of the second degree represents a conic (Art. 175), and also that the three species of conics may be distinguished from each other by the *number* of directions in which lines meeting the curve at infinity may be drawn through any given point (Art. 131, Note), it is easy to find a test that will enable one to distinguish at a glance the kind of conic represented by a given equation.

Let the given equation be

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0. \quad . \quad (1)$$

If this equation be transformed to polar coördinates, the origin being the pole and the x-axis the initial line, so that $x = \rho\cos\theta$ and $y = \rho\sin\theta$, it becomes

$$\rho^2(A\cos^2\theta + 2H\sin\theta\cos\theta + B\sin^2\theta) \\ + 2\rho(G\cos\theta + F\sin\theta) + C = 0. \quad . \quad . \quad (2)$$

One value of ρ, determined by this equation, will be infinite if its direction be such that

$$A\cos^2\theta + 2H\sin\theta\cos\theta + B\sin^2\theta = 0; \quad \text{[Art. 10]}$$

i.e., if $$B\tan^2\theta + 2H\tan\theta + A = 0;$$

i.e., if $$\tan\theta = \frac{-H \pm \sqrt{H^2 - AB}}{A}. \quad . \quad . \quad . \quad (3)$$

Equation (3) shows that $\tan\theta$ will have

two imaginary values, if $H^2 - AB < 0$;

two real and coincident values, if $H^2 - AB = 0$;

two real and distinct values, if $H^2 - AB > 0$.

Therefore, there is no direction, one direction, or there are two directions, respectively, in which a line meeting the curve in an infinitely distant point may be drawn through the origin, according as

$$H^2 - AB \text{ is } < 0, \ = 0, \text{ or } > 0;$$

and hence,

if $H^2 - AB < 0$, equation (1) represents an ellipse,

if $H^2 - AB = 0$, equation (1) represents a parabola,

if $H^2 - AB > 0$, equation (1) represents an hyperbola.

178. Center of a conic section. As already defined (Arts. 111, 117, 120), the center of a curve is a point such that all chords of the curve passing through it are bisected by it. It has also been shown that such a point exists for the ellipse and hyperbola, *i.e.*, that these are **central conics.**

If the equation of the conic is given in the form

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0, \quad . \quad (1)$$

the necessary and sufficient condition that the origin is at the center, is $G = 0$ and $F = 0$.

For if the origin be at the center, and (x_1, y_1) be any given point on the locus of equation (1), then $(-x_1, -y_1)$ must also be on this locus (because these two points are on a straight line through the origin and equidistant from it); hence the coördinates of each of these points satisfy equation (1),

$$i.e., \quad Ax_1^2 + 2Hx_1y_1 + By_1^2 + 2Gx_1 + 2Fy_1 + C = 0, \quad . \quad (2)$$

and $A(-x_1)^2 + 2H(-x_1)(-y_1)$
$$+ B(-y_1)^2 + 2G(-x_1) + 2F(-y_1) + C = 0; \quad (3)$$
and equation (3) may be written thus:
$$Ax_1^2 + 2Hx_1y_1 + By_1^2 - 2Gx_1 - 2Fy_1 + C = 0. \quad . \quad (4)$$

Subtracting equation (4) from equation (2) gives
$$4Gx_1 + 4Fy_1 = 0;$$
i.e.,
$$Gx_1 + Fy_1 = 0. \quad . \quad . \quad . \quad . \quad (5)$$

But equation (5) is to be satisfied by the coördinates x_1 and y_1 of *every* point on the locus of equation (1), and the necessary and sufficient conditions for this are
$$G = 0 \text{ and } F = 0.$$

179. Transformation of the equation of a conic to parallel axes through its center. Let the equation of the given conic be
$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0, \quad . \quad (1)$$
and let the coördinates of its center be α and β. Then to transform equation (1) to parallel axes through the point (α, β) it is only necessary to substitute in that equation $x' + \alpha$ and $y' + \beta$ for x and y. This substitution gives
$$A(x' + \alpha)^2 + 2H(x' + \alpha)(y' + \beta) + B(y' + \beta)^2$$
$$+ 2G(x' + \alpha) + 2F(y' + \beta) + C = 0;$$
i.e.,
$$Ax'^2 + 2Hx'y' + By'^2 + 2x'(A\alpha + H\beta + G)$$
$$+ 2y'(H\alpha + B\beta + F) + A\alpha^2 + 2H\alpha\beta + B\beta^2$$
$$+ 2G\alpha + 2F\beta + C^* = 0. \quad . \quad . \quad . \quad (2)$$

Since α and β are the coördinates of the center (Art. 178),
$$A\alpha + H\beta + G = 0 \text{ and } H\alpha + B\beta + F = 0; \quad . \quad (3)$$

---

* It is to be noted here that the new absolute term, *i.e.*, the term free from x' and y' in equation (2), may be obtained by substituting α and β for x and y in the first member of equation (1).

solving these equations gives

$$\alpha = \frac{BG - FH}{H^2 - AB} \quad \text{and} \quad \beta = \frac{AF - GH}{H^2 - AB}, \quad . \quad . \quad (4)$$

which are the coördinates of the center of the locus of equation (1).

The constant term in equation (2) is,

$$A\alpha^2 + 2H\alpha\beta + B\beta^2 + 2G\alpha + 2F\beta + C,$$

$$= \alpha(A\alpha + H\beta + G) + \beta(H\alpha + B\beta + F) + G\alpha + F\beta + C,$$

$$= G\alpha + F\beta + C, \qquad [\text{by virtue of equations (3)}] \quad . \quad . \quad (5)$$

$$= G\left(\frac{BG - FH}{H^2 - AB}\right) + F\left(\frac{AF - GH}{H^2 - AB}\right) + C, \quad [\text{by equation (4)}]$$

$$= -\frac{ABC + 2FGH - AF^2 - BG^2 - CH^2}{H^2 - AB} = -\frac{\Delta}{H^2 - AB}, \quad (6)$$

wherein

$$\Delta \equiv ABC + 2FGH - AF^2 - BG^2 - CH^2 \text{ (cf. Art. 67).}$$

Equations (4) show that the center of the locus of equation (1) is a definite point, at a finite distance from the origin, if $H^2 - AB \neq 0$, but that the coördinates of this center become infinite if $H^2 - AB = 0$. Hence (cf. Art. 177), while the ellipse and hyperbola each have a definite finite center, the parabola may be regarded as having a center at infinity.

By making use of equations (3) and (5), equation (2) may be written

$$Ax'^2 + 2Hx'y' + By'^2 - \frac{\Delta}{H^2 - AB} = 0; \quad . \quad . \quad (6)$$

hence, if the general equation of an ellipse or hyperbola be transformed to parallel axes through the center of the conic, the coefficients of the quadratic terms remain unchanged, those of the first degree terms vanish, and the new absolute term becomes

$$-\frac{\Delta}{H^2 - AB}.$$

NOTE. Two special cases should be noted:

1) Equation (6) shows that if $\Delta = 0$, the locus of equation (1) consists of two straight lines through the new origin (cf. Art. 67).

2) The point (α, β) is the intersection of the two straight lines

$$Ax + Hy + G = 0 \text{ and } Hx + By + F = 0. \quad \text{(cf. eq. (3) above.)}$$

If $\frac{A}{H} = \frac{H}{B} = \frac{G}{F}$, then these lines are coincident (Art. 38, (β)), and the coördinates α and β become indeterminate. In this case, it may be shown that $\Delta = 0$; that the locus of equation (1) consists of two lines parallel to, on opposite sides of, and equidistant from, the line $Ax + Hy + G = 0$; hence any point of the latter line may be considered as a center, since chords drawn through such a point are bisected by it, *i.e.*, the curve has a *line* of centers. Again, since $H^2 - AB = 0$, this locus may be considered a special form of a parabola.

180. The invariants $A + B$ and $H^2 - AB$. In Art. 175 it was shown that a transformation of coördinates by rotating the axes through an angle θ changes the coefficients of the equation

$$Ax^2 + 2\,Hxy + By^2 + 2\,Gx + 2\,Fy + C = 0, \quad . \quad (1)$$

with the exception of the constant term. It is true, however, that certain functions of these coefficients are not changed by this transformation, *e.g.*, the sum $A + B$ of the coefficients of the x^2 and y^2 terms is the same after transformation as before. If the transformed equation be written

$$A'x^2 + 2\,H'xy + B'y^2 + 2\,G'x + 2\,F'y + C = 0, \quad . \quad (2)$$

wherein, as in Art. 175,

$$A' \equiv A\cos^2\theta + 2\,H\sin\theta\cos\theta + B\sin^2\theta, \quad . \quad (3)$$

$$B' = A\sin^2\theta - 2\,H\sin\theta\cos\theta + B\cos^2\theta, \quad . \quad (4)$$

and $$2\,H' = 2\,H\cos 2\,\theta - (A - B)\sin 2\,\theta, \quad . \quad . \quad . \quad (5)$$

then the addition of equations (3) and (4)

gives $\boldsymbol{A' + B' = A + B}$ (since $\sin^2\theta + \cos^2\theta = 1$). . (6)

Again, $A' - B' = 2H\sin 2\theta + (A - B)\cos 2\theta$ . . (7)

hence

$$(A' - B')^2 + 4H'^2 = \{(A - B)^2 + 4H^2\}(\sin^2 2\theta + \cos^2 2\theta),$$
$$= (A - B)^2 + 4H^2, \qquad (8)$$

i.e., $A'^2 - 2A'B' + B'^2 + 4H'^2 = A^2 - 2AB + B^2 + 4H^2.$

But by (6),

$$A'^2 + 2A'B' + B'^2 = A^2 + 2AB + B^2;$$

hence, by subtraction,

$$\boldsymbol{H'^2 - A'B' = H^2 - AB}, \qquad (9)$$

and the function $H^2 - AB$ is also unchanged by the transformation of coördinates, through the angle θ. Moreover, if a transformation of coördinates to a new origin be performed as in Art. 179, A, B, and H are not changed, nor, therefore, the functions $A + B$ and $H^2 - AB$. Such functions of the coefficients, which do not vary when the transformations of Arts. 175 and 179 are performed, are called **invariants** of the equation for those transformations.

If, as in Art. 175, θ be chosen so that

$$\tan 2\theta = \frac{2H}{A - B}, \qquad (10)$$

then $H' = 0$, and equation (8) becomes

$$-A'B' = H^2 - AB. \qquad (11)$$

Again, from eq. (10), $\sin 2\theta = \dfrac{2H}{\sqrt{(A - B)^2 + 4H^2}}$,

and $\cos 2\theta = \dfrac{A - B}{\sqrt{(A - B)^2 + 4H^2}}$;

hence, equation (8), $A' - B' = \dfrac{2H}{\sin 2\theta}$. . . . (12)

Since $\sin 2\theta$ is positive (Art. 175), therefore the sign of $A' - B'$ is the same as the sign of H.

These results are useful in reducing an equation of a conic to its simplest standard form, as will be illustrated in the following article.

181. To reduce to its simplest standard form the general equation of a conic. *a. Central conic.* The result of Art. 180 enables one to reduce to its simplest form a given equation of the second degree, in which $H^2 - AB \neq 0$, much more easily than by the method of Art. 175. If the equation of the conic,

$$Ax^2 + 2\,Hxy + By^2 + 2\,Gx + 2\,Fy + C = 0, \tag{1}$$

be first transformed to the center of the curve as origin, the resulting equation becomes (Art. 179)

$$Ax^2 + 2\,Hxy + By^2 + C' = 0. \tag{2}$$

If equation (2) be now transformed to axes $O'X''$ and $O'Y''$, making the angle θ with $O'X'$ and $O'Y'$, respectively, such that

$$\tan 2\,\theta = \frac{2\,H}{A-B},$$

it will become (Art. 175)

$$A'x^2 + B'y^2 + C' = 0, \tag{3}$$

wherein the new coefficients are easily determined by the relations

$$C' = G\alpha + F\beta + C$$

$$= -\frac{\Delta}{H^2 - AB},$$

(Art. 179),

$$A' + B' = A + B,$$

and $-A'B' = H^2 - AB$

(Art. 180).

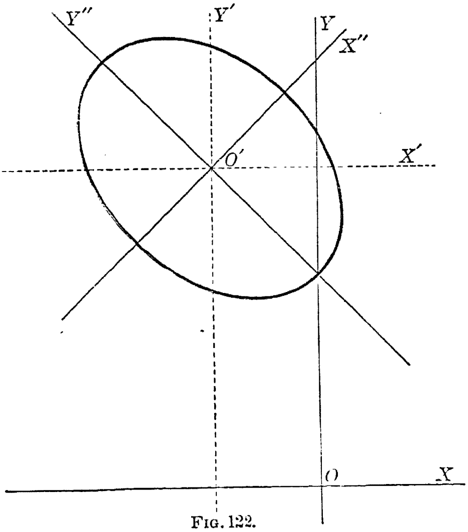


FIG. 122.

EXAMPLE. Suppose the given equation to be

$$3\,x^2 + 2\,xy + 3\,y^2 - 16\,y + 20 = 0, \tag{4}$$

in which $A = 3$, $H = 1$, $B = 3$, $G = 0$, $F = -8$, and $C = 20$.

Then $H^2 - AB = -8$, and the locus is an ellipse.

The coördinates of the center are $\alpha = -1$, $\beta = 3$.

Therefore, $C' = G\alpha + F\beta + C = -4$; $A' + B' = 6$, $-A'B' = -8$;

and, since A' is larger than B', H being positive (Art. 180),

hence $$A' = 4,\ B' = 2;$$

while $\tan 2\theta = \infty$, and therefore $\theta = 45°$. The transformed equation is therefore

$$4x^2 + 2y^2 - 4 = 0,$$

i.e., $$\frac{x^2}{1} + \frac{y^2}{2} = 1, \quad . \quad . \quad . \quad (5)$$

when referred to the axes $O'X''$, $O'Y''$; and the locus is approximately as given in Fig. 122.

b. Non-central conic. If $H^2 - AB = 0$, the relations of equations (6) and (11), Art. 180, may still be used to simplify the reduction of equation (1) to the standard form for the equation of a parabola, if, as in Art. 176, the xy-term be removed first. In this case, however, a better method of reduction is as follows:

Since the first three terms of equation (1) form a perfect square, that equation may be written

$$(\sqrt{A}x + \sqrt{B}y)^2 + 2Gx + 2Fy + C = 0 \quad . \quad . \quad . \quad (6)$$

wherein the sign of the $\sqrt{B}$ is the same as that of H.

Equation (2) may now be transformed to new axes OX' and OY', which are so chosen that the equation of OX' referred to the given axes shall be

$$\sqrt{A}\,x + \sqrt{B}\,y = 0;$$

hence, if θ be the angle between OX and OX', then

$$\tan\theta = -\frac{\sqrt{A}}{\sqrt{B}}, \text{ whence } \sin\theta = \frac{-\sqrt{A}}{\sqrt{A+B}} \text{ and } \cos\theta = \frac{\sqrt{B}}{\sqrt{A+B}} \quad . \quad (7)$$

Equation (7) shows that θ is *negative* (if the positive value of $\sqrt{A+B}$ be used), and acute or obtuse according as $\sqrt{B}$ is positive or negative. The formulas for transforming to the new axes are (cf. Art. 72)

$$x = \frac{\sqrt{B}}{\sqrt{A+B}}x' + \frac{\sqrt{A}}{\sqrt{A+B}}y' \text{ and } y = \frac{-\sqrt{A}}{\sqrt{A+B}}x' + \frac{\sqrt{B}}{\sqrt{A+B}}y'. \quad . \quad (8)$$

Substituting these values for x and y in equation (6), it becomes

$$(A+B)y'^2 + 2\frac{G\sqrt{B} - F\sqrt{A}}{\sqrt{A+B}}x' + 2\frac{G\sqrt{A} + F\sqrt{B}}{\sqrt{A+B}}y' + C = 0. \quad . \quad (9)$$

By dividing equation (9) by $(A + B)$, completing the square of the y'-terms, and transposing, it may be written in the form

$$\left\{y' + \frac{G\sqrt{A} + F\sqrt{B}}{(A+B)^{\frac{3}{2}}}\right\}^2 = -2\frac{G\sqrt{B} - F\sqrt{A}}{(A+B)^{\frac{3}{2}}}\left\{x' + \frac{(G\sqrt{A} + F\sqrt{B})^2 - C(A+B)^2}{2(A+B)^{\frac{3}{2}}(G\sqrt{B} - F\sqrt{A})}\right\}. \quad (10)$$

Comparing equation (10) with equation [42] (Art. 106), it is seen that the length of the latus rectum, as well as the coördinates of the vertex and focus (with reference to the axes OX' and OY'), and other important facts, may be read directly from the equation.

The advantage of equation (10), over that resulting from the reduction of Ex. 2, Art. 176, is that, in connection with equation (7), it gives all the facts necessary for the immediate location of the curve, and gives those facts in terms of the coefficients of the original equation.

EXAMPLE. Let it be required to determine the position and parameter of the parabola represented by the equation

$$9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0.$$

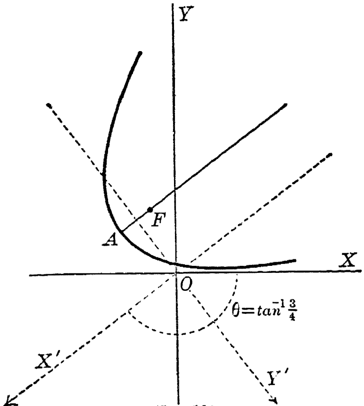


FIG. 123.

The given equation may be written as

$$(3x - 4y)^2 - 18x - 101y + 19 = 0.$$

If the line $3x - 4y = 0$ be chosen as x'-axis, then $\tan\theta = \frac{3}{4}$, whence $\sin\theta = -\frac{3}{5}$, and $\cos\theta = -\frac{4}{5}$. The formulas of transformation then are:

$$x = \frac{-4x' + 3y'}{5} \text{ and } y = -\frac{3x' + 4y'}{5}.$$

Substituting these values in equation (1), it becomes

$$25y'^2 + 70y' = -75x' - 19;$$

this equation may be written

$$(y' + \tfrac{7}{5})^2 = -3(x' - \tfrac{2}{5}),$$

which shows that the latus rectum is 3, and the coördinates of the vertex and focus (with reference to the new axes) are, respectively, $\frac{2}{5}, -\frac{7}{5}$ and $-\frac{7}{20}, -\frac{7}{5}$. It also shows that the axis of the curve is parallel to the negative end of the x'-axis.

Recalling the remark about the angle θ determined by equations (7) above, it is seen that the geometric representation of the above equation is shown in Fig. 123.

182. Summary. It has been shown in the preceding articles that every equation of the second degree in two variables represents a conic section, whether the axes are oblique or rectangular; and that its species and position depend upon the values of the coefficients of the equation. The various criteria of the nature of the conic represented by such an equation, in *rectangular coördinates*, appear in the following table:

The General Equation of the Second Degree

$$Ax^2 + 2\,Hxy + By^2 + 2\,Gx + 2\,Fy + C = 0$$

$$\Delta \equiv ABC + 2\,FGH - AF^2 - BG^2 - \cancel{HC^2}\ CH^2$$

I. $H^2 - AB < 0$. The ellipse.

(1) if $A = B$, and $H = 0$, a circle.

(2) if Δ is $+$, imaginary.

(3) if Δ is $-$, real.

(4) if Δ is 0, a pair of imaginary straight lines; or, a point.

II. $H^2 - AB = 0$. The parabola.

(1) if H is $+$, principal axis is the new y-axis.

(2) if H is $-$, principal axis is the new x-axis.

(3) if Δ is 0, pair of parallel straight lines, which are real and different, real and coincident, or imaginary, according as $G^2 - AC >, =,$ or < 0.

III. $H^2 - AB > 0$. The hyperbola.

(1) if $A = -B$, a rectangular hyperbola.

(2) if Δ is $+$, principal axis is the new y-axis.

(3) if Δ is $-$, principal axis is the new x-axis.

(4) if Δ is 0, a pair of real intersecting straight lines.

NOTE. The above results have not all been shown, but are easily deduced from the work already given. Thus the locus of equation (3), Art. 181, if an ellipse, is imaginary if C' is $-$; but, by equation (6), Art. 179, C' is $-$ if Δ is $+$; hence the test I (2), given above. And so for the other tests, which the student should verify. The angle θ which the new axes make with the old, respectively, is chosen as in Art. 175, 2θ being taken always positive and not greater than 180°.

183. The equation of a conic through given points. The general equation of a conic may be written

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0, \quad . \quad (1)$$

and contains five parameters, the five ratios between the coefficients A, H, B, G, F, C. Since five equations, or conditions, will determine those parameters, in general five points will determine a conic. That is, in general, *a conic may be made to pass through five, and only five, given points.*

If, however, the conic is to be a parabola, one equation is given; viz. $H^2 - AB = 0$, hence only four additional conditions are needed. In general, *a parabola may be made to pass through four points, only.*

A circle has two conditions given, viz. $A = B$, $H = 0$; therefore, in general, *a circle may be made to pass through three points, only.*

A pair of straight lines has one condition given, $\Delta = 0$; therefore, in general, *a pair of straight lines may be made to pass through four points, only.*

The method to be followed in obtaining the equation of the required conic has been used in Art. 80, and may be indicated for finding the equation of the parabola through four given points,

$P_1 \equiv (x_1, y_1)$, $P_2 \equiv (x_2, y_2)$, $P_3 \equiv (x_3, y_3)$, and $P_4 \equiv (x_4, y_4)$.

The equation must be of the form (1),

therefore,
$$Ax_1^2+2\,Hx_1y_1+By_1^2+2\,Gx_1+2\,Fy_1+C=0,$$
$$Ax_2^2+2\,Hx_2y_2+By_2^2+2\,Gx_2+2\,Fy_2+C=0,$$
$$Ax_3^2+2\,Hx_3y_3+By_3^2+2\,Gx_3+2\,Fy_3+C=0,$$
$$Ax_4^2+2\,Hx_4y_4+By_4^2+2\,Gx_4+2\,Fy_4+C=0\;;$$
also,
$$H^2-AB=0.$$

The required ratios between the coefficients of equation (1) may be found from these equations.

EXAMPLES ON CHAPTER XII

Without transforming the equations to other axes, find the center or the vertex, the axes, and the nature of the following conics:

1. $x^2+5\,xy+y^2+8\,x-20\,y+15=0$;

2. $(x-y)^2+2\,x-y=1$;

3. $3\,x^2+2\,y^2-2\,x+y-1=0$;

4. $3\,x^2-8\,xy-3\,y^2+x+17\,y-10=0$;

5. $4\,x^2-4\,xy+y^2+4\,ax-2\,ay=0$;

6. $5\,x^2+2\,xy+5\,y^2=0$;

7. $3\,x^2+3\,y^2+11\,x-5\,y+7=0$;

8. $x^2+2\,xy-y^2+8\,x+4\,y-8=0$;

9. $y^2-xy-6\,x^2+y-3\,x=0$;

10. $y^2-xy-5\,x+5\,y=0$.

Trace the following conics:

11. $3\,x^2+2\,xy+3\,y^2-16\,y+23=0$;

12. $4\,x^2+9\,y^2+8\,x+36\,y+4=0$;

13. $3\,x^2-3\,y^2+8\,xy-10\,y+6\,x+5=0$;

14. $(x-y)(x-y-6)+9=0$.

15. What conic is determined by the points $(0, 3)$, $(1, 0)$, $(2, 1)$, $({}^-1, {}^-3)$, and $(3, {}^-3)$?

16. Find the equation of the parabola through the points $(3, 2)$, $(1, \frac{2}{9})$, $({}^-6, 8)$, and $({}^-2, \frac{8}{9})$.

17. Find the equation of the conic through the points $(9, 2)$, $(6, 3)$, $(3, 2)$, $(1, {}^-2)$, $(2, 1)$.

CHAPTER XIII

HIGHER PLANE CURVES

184. Definitions. A curve, in Cartesian coördinates, whose equation is reducible to a finite number of terms, each involving only positive integer powers of the coördinates, is called an **algebraic curve**; all other curves are called **transcendental curves.**

Algebraic curves the degree of whose equations exceeds two, and all transcendental curves, are (if they lie wholly in a plane) called **higher plane curves.** On account of their great historical interest, and because of their frequent use in the Calculus, a few of these curves will be examined in the present chapter.

I. ALGEBRAIC CURVES

185. The cissoid of Diocles.* The cissoid may be defined as follows: let $OFAK$ be a fixed circle of radius a, OA a diameter, AT a tangent; draw any line as OQS through O, meeting the circle in Q and the tangent in S, and on this line lay off the distance $OP = QS$: the locus of the point P, as the line OS revolves about O, is the **cissoid.***

---

* This curve was invented, by a Greek mathematician named Diocles, for the purpose of solving the celebrated problem of the insertion of two mean proportionals between two given straight lines. The solution of this problem carries with it the solution of the even more famous Delian problem of constructing a cube whose volume shall be equal to two times the volume of a given cube. For, let a be the edge of the given cube; construct the two mean proportionals x and y between a and $2a$; then $a : x :: x : y :: y : 2a$, whence $x^3 = 2 \cdot a^3$, *i.e.*, x is the edge of the required cube. If $a = 1$, then $x = \sqrt[3]{2}$, hence the insertion of two mean proportionals enables one to construct a line equal to the cube root of 2. The cissoid may also be employed to construct a line equal to the cube root of any given number (see Klein, Elementargeometrie, S. 35, or the English translation by Professors Beman and Smith).

It is not positively known just when Diocles lived; it is very probable, however, that it was in the last half of the second century B.C.

From this definition, the equation of the cissoid, referred to the rectangular axes OX and OY, is readily derived.

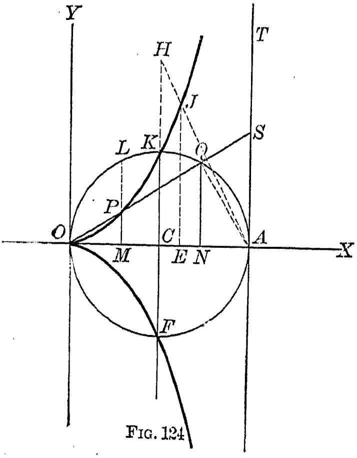


FIG. 124

Let the coördinates of P be x and y, and let C be the center of the circle so that

$$OC = CA = CK = a.$$

Since triangles OMP and ONQ are similar,

$$\therefore MP : OM :: NQ : ON, \quad (1)$$

and since $OP = QS$, therefore

$$NA = OM = x; \text{ moreover,}$$

$$\overline{NQ}^2 = ON \cdot NA = (2a - x)x.$$

Substituting these values in equation (1) gives

$$y : x :: \sqrt{(2a - x)x} : (2a - x), \quad . \quad . \quad . \quad (2)$$

whence

$$y^2 = \frac{x^3}{2a - x}, \quad . \quad . \quad . \quad (3)$$

which is the required rectangular equation of the cissoid.

The definition of the cissoid, as well as the equation just derived, shows that the curve is symmetric with regard to the x-axis; that it lies wholly between the y-axis and the line $x = 2a$; that it passes through the extremities F and K of the diameter perpendicular to OA; and that it has two infinite branches to each of which the line $x = 2a$ is an asymptote.

---

* Diocles named his curve "cissoid" (from a Greek word meaning "ivy," because of its resemblance to a vine climbing upwards. The name "cissoid" is sometimes, though rarely, applied to other curves which are generated as stated in the definition given above, except that some other basic curve is employed instead of a circle. For other, but equivalent, definitions of the cissoid see Note 3, below.

NOTE 1. The polar equation of the cissoid referred to the initial line OX, and pole O, is also easily found. Let the polar coördinates of P be ρ and θ; then,

$$\rho = OP = QS = OS - OQ, \qquad (4)$$

but $$OS = 2a \sec\theta, \text{ and } OQ = 2a\cos\theta,$$

$$\therefore \quad \rho = 2a\sec\theta - 2a\cos\theta = 2a(\sec\theta - \cos\theta),$$

i.e., $$\rho = 2a\tan\theta\sin\theta, \qquad (5)$$

which is the polar equation sought.

NOTE 2. To "duplicate the cube" by means of the cissoid,* extend CK to H, making $HK = CK = a$, draw the line HA cutting the cissoid in J, and draw the ordinate EJ. Since $CH = 2\,CA$, therefore $EJ = 2\,EA$; but from equation (3),

$$\overline{EJ}^2 = \frac{\overline{OE}^3}{EA} = \frac{\overline{OE}^3}{\frac{1}{2}EJ};$$

$$\therefore \quad \overline{EJ}^3 = 2\,\overline{OE}^3. \qquad (6)$$

Now let m be the edge of any given cube, and let it be required to construct a line n such that the cube on n shall be equal to the double of the cube on m. Construct n so that

$$OE : EJ :: m : n;$$

then $$\overline{OE}^3 : \overline{EJ}^3 = m^3 : n^3,$$

and, since $\overline{EJ}^3 = 2 \cdot \overline{OE}^3$, therefore $n^3 = 2\,m^3$.

NOTE 3. The cissoid may also be defined in either of the following ways: (1) as the locus of the point (P) in which the chord OQS intersects that ordinate (ML) of the circle which is equal to NQ; and (2) as the locus of the foot of the perpendicular let fall from the vertex of a parabola upon a tangent. The derivation of the equation of the curve based upon these definitions is left as an exercise for the student.

---

* To insert two mean proportionals between two given lines by means of the cissoid. See Cantor, Geschichte der Mathematik, Bd. I., S. 339.

For Newton's method of drawing the cissoid by continuous motion, see Salmon's Higher Plane Curves, p. 183, or Lardner's Algebraic Geometry, p. 196.

186. The conchoid of Nicomedes.* The conchoid may be defined as follows: Let $PRP'Q$ be a fixed circle of radius a whose center S moves along a fixed straight line OX; let LK be a straight line drawn through a fixed point A and the center S of this moving circle, and let P and P' be the intersections of this line and the circle; then the locus traced by P (and by P') as S moves along OX is a **conchoid.**

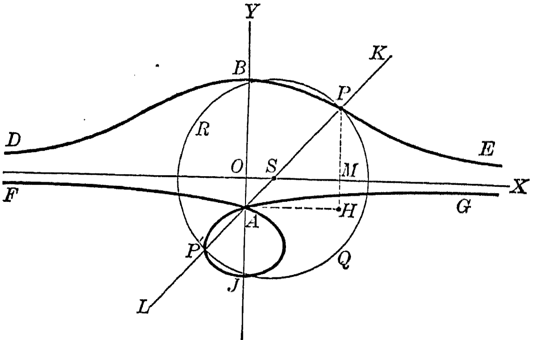


Fig. 125

This definition may also be stated thus: If A is a fixed point, OX a fixed line, and S the point in which OX is intersected by a line LK revolving about A, then the locus of a point P on LK, so taken that SP is always equal to a given constant a, is a conchoid.

The fixed point A is called the **pole,** the constant parameter a the **modulus,** and the fixed line OX the **directrix** of the conchoid.

---

* The conchoid was invented by a Greek mathematician named Nicomedes, probably in the second century B.C. Like the cissoid, it was invented for the purpose of solving the famous problem of the "duplication of the cube"; it is, however, easily applied to the solution of the related, and no less famous, problem of the trisection of a given angle (see Note 3, below).

To derive the rectangular equation of the conchoid draw AOY perpendicular, and AH parallel, to OX, and let $OA = c$; let $P \equiv (x, y)$ be any position of the generating point, and draw the ordinate HMP; then, from the similar triangles AHP and SMP,

$$AH : HP :: SM : MP,$$

i.e.,
$$x : y + c :: \sqrt{a^2 - y^2} : y;$$

$$\left[\text{since } SM = \sqrt{\overline{SP}^2 - \overline{MP}^2} = \sqrt{a^2 - y^2}\right],$$

whence
$$x^2y^2 = (y + c)^2(a^2 - y^2),$$

which is the equation sought.

The definition of the conchoid, as well as the equation just derived, shows that the curve is symmetric with regard to the y-axis; that it lies wholly between the two lines $y = a$ and $y = -a$; and that it has four infinite branches to each of which the x-axis is an asymptote.*

NOTE 1. The polar equation of the conchoid. Let A be the pole, AY the initial line, and $P \equiv (\rho, \theta)$ (or P') any position of the generating point; then

$$\rho = AP = AS \pm SP = OA \cdot \sec\theta \pm SP,$$

i.e.,
$$\rho = c \sec\theta \pm a,$$

which is the desired equation.

NOTE 2. The conchoid may also be readily constructed by continuous motion as follows: By means of a slot in a ruler, fitting over a pin at A, the motion of the line LK is properly controlled; if now a guide pin at S, and a tracing point at P, be attached to this ruler, then the point P will trace out the conchoid when the guide point S is moved along the line OX.

NOTE 3. By means of a conchoid, any given angle may be trisected.† Let ABC be any angle, on one side (BA) take any distance, as BH, and draw OHX perpendicular to the other side of the angle (BC); then lay off $OK = 2\,BH$, and construct the conchoid KEF with B as pole and $BH = \frac{1}{2}\,OK$ as modulus, and OX as directrix. Draw HL parallel to BC and connect B with L, then the angle $LBC = \frac{1}{3}\,ABC$; for, join D, the middle point of ML, to H, then $ML = OK = 2\,BH = 2\,HD$, and the three angles marked α are all equal, as are also the two marked β; moreover, $\beta = 2\,\alpha$, being the exterior angle of the triangle HLD, which proves that angle $LBC = \frac{1}{3}\,ABC$.

---

* It is evident that, if $AO < OB$, *i.e.*, if $c < a$, the curve has an oval below A as shown in Fig. 2; if $c = a$, this oval closes up to a point; and if $c > a$, both parts of the curve lie wholly above A.

† For the insertion of two mean proportionals between two given lines by means of the conchoid, see Cantor, Geschichte der Mathematik, Bd. I., S. 336.

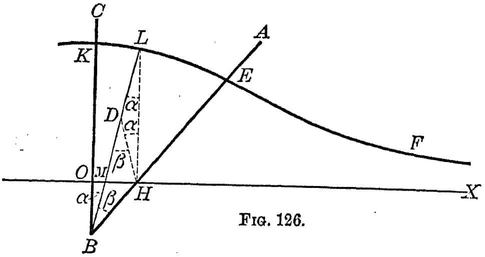


Fig. 126.

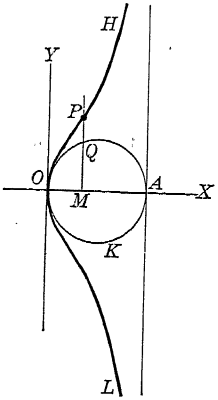


Fig. 127.

187. The witch of Agnesi.* The witch may be defined as follows: Let $OKAQ$ be a given fixed circle of radius a, OA a diameter, and Q any point on the circle; if now the ordinate MQ be produced to P, so that

$$MQ : MP :: MA : OA, \quad . \quad . \quad (1)$$

then the locus of P, as Q moves around the circle, is the **witch.** To derive the rectangular equation of the witch, let $P \equiv (x, y)$ be any point on the curve; then, since

$$MQ = \sqrt{OM \cdot MA} = \sqrt{x(2a - x)},$$

---

* The witch was invented by Donna Maria Gaetana Agnesi (1718–1799) an Italian lady who was appointed professor of mathematics at the University of Bologna, in 1750.

substituting in equation (1) gives

$$\sqrt{x(2a-x)} : y :: (2a-x) : 2a, \quad . \quad . \quad . \quad (2)$$

i.e.,

$$y^2 = \frac{4a^2x}{2a-x}, \quad . \quad . \quad . \quad (3)$$

which is the equation sought.

The definition of the witch, as well as the equation just derived, shows that the curve is symmetrical with regard to the x-axis; that it lies wholly between the y-axis and the line $x = 2a$; and that it has two infinite branches to each of which the line $x = 2a$ is an asymptote.

188. The lemniscate of Bernouilli.* The lemniscate may be defined as follows: let $LTARNA'K$ be a rectangular hyperbola, O its center, OX and OY its axes, and TE a tangent to the curve at any point T. Also let OG be a perpendicular from the center upon this tangent, and let P be the point of their intersection; then the locus of P as T moves along the hyperbola is called the **lemniscate.**

To derive the rectangular equation of this curve, let $OA = a$, and let the coördinates of T be x_1 and y_1; then the equation of the tangent TE is

$$x_1x - y_1y = a^2, \quad . \quad . \quad . \quad (1)$$

hence the equation of OG, the perpendicular upon this tangent (Art. 62), is

$$x_1y + y_1x = 0. \quad . \quad . \quad . \quad (2)$$

---

* The lemniscate was invented by Jacques Bernouilli (1654–1705), a noted Swiss mathematician and professor in the University of Basle. It is, however, only a special case of the Cassinian ovals; viz., of the locus of the vertex of a triangle whose base is given in length and position, and the product of whose other two sides is a constant. See Salmon's Higher Plane Curves, p. 44, Gregory's Examples, or Cramer's Introduction to the Analysis of Curves.

Regarding equations (1) and (2) as simultaneous, the x and y involved are the coördinates of the point P; moreover, since the point $T \equiv (x_1, y_1)$ is on the hyperbola, therefore

$$x_1^2 - y_1^2 = a^2. \quad . \quad . \quad . \qquad (3)$$

Eliminating x_1 and y_1 between equations (1), (2), and (3) gives

$$(x^2 + y^2)^2 = a^2(x^2 - y^2), \quad . \quad . \quad . \qquad (4)$$

which is, therefore, the equation sought.

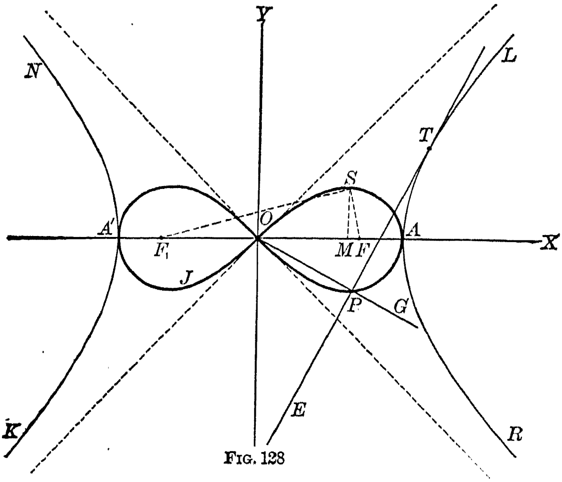


FIG. 128

The definition of the lemniscate, as well as the equation just derived, shows that the curve is symmetrical with regard to both coördinate axes; that it lies wholly between the two lines whose equations are $x = -a$ and $x = +a$; that it passes through the origin and the two points $(-a, 0)$ and $(+a, 0)$; and that y is never larger than x; hence the lemniscate is a limited closed curve as represented in Fig. 128.

NOTE 1. The polar equation of the lemniscate is easily derived from equation (4) if the x-axis be chosen as initial line and the origin as pole;

for then $x = \rho\cos\theta$ and $y = \rho\sin\theta$, and equation (4) at once reduces to

$$\rho^2 = a^2(\cos^2\theta - \sin^2\theta) = a^2\cos 2\theta, \quad . \quad . \quad . \quad (5)$$

which is therefore the required polar equation of the lemniscate.

Equation (5) shows that: when $\theta = 0$, $\rho = \pm a$; when $\theta < 45°$, ρ has two equal but opposite values, each of which is smaller than a; when $\theta = 45°$, $\rho = 0$, *i.e.*, the angle which the curve makes with the initial line is 45°; when $45° < \theta < 135°$, ρ is imaginary; when $135° < \theta < 180°$, ρ has two equal but opposite values, each of which is smaller than a; and when $\theta = 180°$, $\rho = \pm a$. The curve, therefore, consists of two ovals meeting in O, each lying in the same angle between the asymptotes of the hyperbola as does the corresponding branch of that curve, and these asymptotes are tangent to the lemniscate at the point O.

NOTE 2. If the two points F_1 and F be so located that

$F_1O = OF = \frac{a}{2}\sqrt{2}$, and if $S \equiv (x, y)$ be any point on the lemniscate,

then $\qquad F_1S = \sqrt{\overline{F_1M}^2 + \overline{MS}^2} = \sqrt{\left(\frac{a}{2}\sqrt{2} + x\right)^2 + y^2},$

and $\qquad FS = \sqrt{\left(\frac{a}{2}\sqrt{2} - x\right)^2 + y^2},$

hence $F_1S \cdot FS = \sqrt{\left(\frac{a}{2}\sqrt{2} + x\right)^2 + y^2} \cdot \sqrt{\left(\frac{a}{2}\sqrt{2} - x\right)^2 + y^2}$

$$= \sqrt{(x^2 + y^2)^2 - a^2(x^2 - y^2) + \frac{a^4}{4}} = \frac{a^2}{2}, \text{ [by eq. (4)]},$$

i.e., $\quad F_1S \cdot FS = \frac{a^2}{2}.$

Hence the lemniscate may be defined as the locus of a point which moves so that the product of its distances from two fixed points is constant, and equal to the square of half the distance between the fixed points (cf. foot-note, p. 315).

This definition of the curve easily leads to the equation already derived; it also enables one to readily construct the curve thus: with F as center, and any convenient radius FS, describe an arc; then, with F_1 as center, and a third proportional to FS and OF as radius, describe another arc cutting the first in S; this intersection S is a point on the locus, and as many points as desired may be constructed in the same way.

189*a*. The limaçon of Pascal.[*] The limaçon may be defined as generated from a circle by adding a constant length to each of the radii vectores drawn from a point on its circumference as origin, — proper account being taken of negative radii vectores.[†] *E.g.*, let OLA_1N be a given circle of radius a, O any point on it, $A_1A = k$ any constant; then if any radius vector as OP_1 be drawn from O, and $P_1P = A_1A = k$ be added to it, then P is a point on the limaçon; and as P_1 is made to describe a circle, P will trace the limaçon.

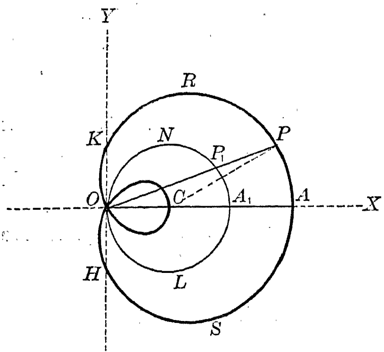


FIG. 129

The polar equation of the curve is at once written down from this definition; for, if the diameter OCX be taken as initial line, then the polar equation of the circle is

$$\rho = 2\,a\cos\theta, \qquad . \quad . \quad . \quad (1)$$

whence the polar equation of the limaçon is

$$\rho = 2\,a\cos\theta + k. \qquad . \quad . \quad . \quad (2)$$

If k be taken equal to a, the radius of the given circle, this equation may be written in the more common form

$$\rho = a(1 + 2\cos\theta). \qquad . \quad . \quad . \quad (3)$$

---

[*] This curve was invented and named by Blaise Pascal (1623–1662), a celebrated French geometrician and philosopher. It is, however, a special case of the so-called *Cartesian ovals*.

[†] The limaçon may also be defined as the locus of the intersection of the two lines OP and CP which are so related during their revolution about O and C, respectively, that the angle XCP is always equal to $\frac{3}{2}$ times the angle XOP. This definition easily leads to the polar equation already derived.

The definition of the limaçon, as well as the equation just derived, shows that the curve is symmetrical with regard to the initial line, and that it has the form shown in Fig. 129.

NOTE. The rectangular equation of the limaçon for which $k = a$ is easily derived from equation (3). Choosing the initial line and a perpendicular to it through O as rectangular axes, so that $x = \rho \cos \theta$, and $y = \rho \sin \theta$, equation (3) becomes

$$\sqrt{x^2 + y^2} = a + 2a \cdot \frac{x}{\sqrt{x^2 + y^2}}. \quad . \quad . \quad . \quad . \quad (4)$$

Rationalizing equation (4) gives

$$(x^2 + y^2 - 2ax)^2 = a^2(x^2 + y^2), \quad . \quad . \quad . \quad . \quad (5)$$

which is the usual form for the rectangular equation of the limaçon.

189*b*. The cardioid. The cardioid may be defined as a special case of the limaçon; viz., it is a limaçon in which the constant k, which is added to each of the radii vectores, is taken equal to the diameter of the fundamental circle. If in the equation of the limaçon [Art. 189*a*, equation (2)] the constant k be taken equal to $2a$, that equation becomes

$$\rho = 2a(1 + \cos \theta), \quad . \quad . \quad . \quad (1)$$

which is the polar equation of the cardioid.

The more usual form in which the equation of the cardioid is written is

$$\rho = 2a(1 - \cos \theta), \quad . \quad . \quad . \quad (2)$$

but this amounts merely to turning the figure through 180° in its own plane.

NOTE 1. The rectangular equation of the cardioid is obtained as in Art. 189 *a*.

It is $(x^2 + y^2 + 2ax)^2 = a^2(x^2 + y^2)$. (3)

The curve represented by equations (2) and (3) has the form shown in Fig. 130.

The cardioid is usually defined as the locus traced by a point on a given circle AKA_1L, which rolls on an equal but fixed circle OMA_1H. This definition also leads to equations (2) and (3) already derived.

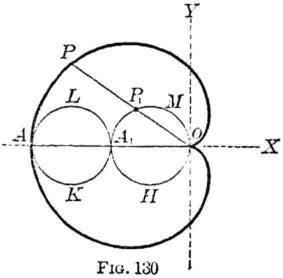


FIG. 130

190. The Neilian, or semi-cubical, parabola.* This curve may be defined as follows: let $HTASKL$ be a given parabola whose equation is

$$y^2 = 4px; \quad . \quad . \quad . \quad (1)$$

let TMS be any double ordinate of the curve, TT_1 a tangent at the point $T \equiv (x_1, y_1)$, and AQ a perpendicular from the vertex upon this tangent; if QA intersects TS in P, then the locus of P as T moves along the parabola is called a **semi-cubical** or **Neilian** parabola.

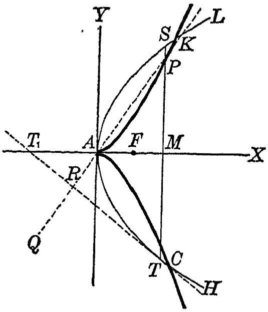


FIG. 131

Its rectangular equation is derived as follows: the equation of TT_1 is

$$y_1y = 2p(x + x_1), \quad . \quad . \quad . \quad (2)$$

hence the equation of AQ is

$$y = -\frac{y_1}{2p}x. \quad . \quad . \quad . \quad (3)$$

The equation of TS is

$$x = x_1. \quad . \quad . \quad . \quad (4)$$

If now equations (3) and (4) be regarded as simultaneous, then x and y are the coördinates of the point P in which the two lines intersect, and if x_1 and y_1 be eliminated by means of the equation

$$y_1^2 = 4px_1, \quad . \quad . \quad . \quad (5)$$

an equation connecting x and y is obtained.

---

* This curve is historically interesting, because it is the first one which was *rectified*, *i.e.*, it is the first one the length of an arc of which was expressed in rectilinear units. This celebrated rectification was performed, without the aid of the modern Calculus methods, by William Neil, a pupil of Wallis (see Cantor, Geschichte der Mathematik, Bd. II., S. 827), in 1657, and is, for that reason, called the *Neilian* parabola. It is also called the *semi-cubical* parabola because its equation may be written in the form $y = ax^{\frac{3}{2}}$.

Substituting for x_1 and y_1, in equation (5), their values in terms of x and y as found from equations (3) and (4), gives

$$\frac{4p^2}{x^2}y^2 = 4px,$$

i.e.,

$$y^2 = \frac{x^3}{p}, \quad . \quad . \quad . \quad (6)$$

which is the equation sought.

This equation shows that the curve passes through the origin and is symmetrical with regard to the x-axis; that it lies wholly on the same side of the y-axis as does the given parabola; and that it has two infinite branches.

II. TRANSCENDENTAL CURVES.*

191. The cycloid.† The cycloid ($OPKA$) is the path traced by a point P on the circumference of a circle ($HNSP$)

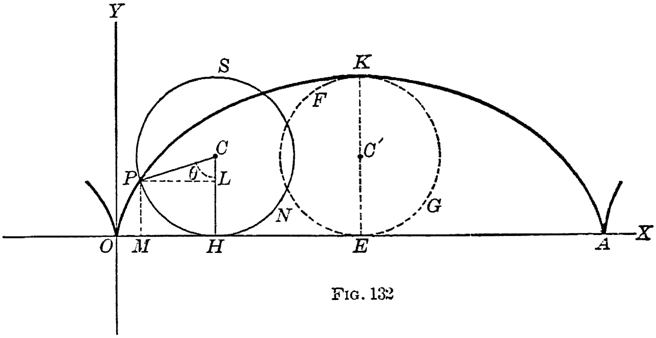


FIG. 132

---

* A few very common transcendental curves have already been examined in Chapter III; among these are the curve of sines, the curve of tangents, and the logarithmic curve.

† Because of the elegance of its properties, and because of its numerous applications in mechanics, the cycloid is the most important of the transcendental curves. It has the added historical interest of being the second curve that was rectified (cf. Art. 190, foot-note). Its rectification was first accomplished by Sir Christopher Wren (1632–1723) and published by him in 1673.

which rolls, without sliding, upon a fixed right line (OX). The point P is called the **generating point**; the circle $PHNS$, the **generating circle**; the points O and A, the **vertices**; the line EK, perpendicular to OA at its middle point, the **axis**; and the line OA, the **base** of the cycloid.

To derive the rectangular equation of the cycloid let a be the radius of the generating circle, and OX the fixed straight line on which it rolls; also let P be the generating point, and let PNS be any position of the generating circle. Draw the radius CP, the ordinate MP, the line PL parallel to OX, and the radius OH to the point of contact of the generating circle and the line OX. Let OX and OY (the perpendicular to it through O) be chosen as axes, and let θ be the angle PCH.

Then, if $P \equiv (x, y)$,

$$\begin{aligned} x = OM &= OH - MH \\ &= OH - PL \\ &= a\theta - a\sin\theta, \quad [\text{since } OH = \text{arc } PH = a\theta]. \end{aligned}$$

i.e., $$x = a(\theta - \sin\theta). \quad . \quad . \quad . \qquad (1)$$

Similarly, $$y = a(1 - \cos\theta). \quad . \quad . \quad . \qquad (2)$$

Solving equation (2) for θ gives

$$\cos\theta = \frac{a-y}{a},$$

i.e., $$\theta = \cos^{-1}\left(\frac{a-y}{a}\right) = \text{vers}^{-1}\left(\frac{y}{a}\right);$$

and substituting this value of θ in equation (1) gives

$$x = a \,\text{vers}^{-1}\left(\frac{y}{a}\right) - \sqrt{2ay - y^2}, \quad . \quad . \quad . \qquad (3)$$

which is the rectangular equation sought.

NOTE 1. It is usually simpler to regard equations (1) and (2) together as representing the cycloid; θ is then the independent variable, while x and y are both functions of it.

NOTE 2. The cycloid belongs to the kind of curves called **roulettes.** These curves are generated by a point which is invariably connected with a curve which rolls, without sliding, upon a given fixed curve.

If both the rolling and the fixed curves are *circles*, then the curve generated is designated by the general name of **trochoid.** If the generating point is on the *circumference* of the rolling circle, and this circle rolls on the *outside* of a fixed circle, then the curve described is called an **epicycloid**; but if it rolls on the *inside* of the fixed circle, the generated curve is called a **hypocycloid.** The cycloid may be regarded either as an epicycloid or a hypocycloid, for which the fixed circle has its center at infinity and an infinite radius.

192. The hypocycloid. Let the hypocycloid $APRST\cdots$ be traced by the point P on the circumference of the circle PQR, whose radius is b, and which rolls on the inside of the fixed circle AQE, whose radius is a. Also let $P \equiv (x, y)$ be any position of the generating point. Draw the line $OO'Q$, the ordinates HO' and MP, the radius $O'P$, and the

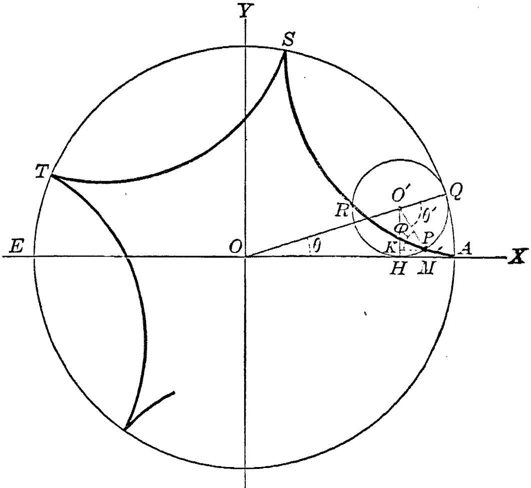


FIG. 133

line KP parallel to OA, where A is the point with which P coincided when in its initial position. Let OAX and OY, the perpendicular to it through O, be chosen as coördinate axes; also let the angles AOQ, $PO'Q$ and $O'PK$ be designated, respectively, by θ, θ' and ϕ.

Then $OM = OH + HM = OH + KP$

$$= OO' \cos\theta + PO' \cos\phi$$

$$= OO' \cos\theta + PO' \cos(\theta' - \theta),$$

[since $\phi = \theta' - \theta$]

i.e., $$x = (a - b)\cos\theta + b\cos(\theta' - \theta). \quad . \quad . \quad . \quad (1)$$

But since arc AQ = arc PQ, therefore $a\theta = b\theta'$, whence $\theta' = \frac{a}{b}\theta$, and equation (1) becomes

$$x = (a - b)\cos\theta + b\cos\frac{(a-b)\theta}{b}. \quad . \quad . \quad . \quad (2)$$

Similarly, $$y = (a - b)\sin\theta - b\sin\frac{(a-b)\theta}{b}. \quad . \quad . \quad . \quad (3)$$

Equations (2) and (3) are together the equations of the hypocycloid. A single equation representing the same curve may be found, as in the case of the cycloid (Art. 191), by eliminating θ between equations (2) and (3).

NOTE. If the radii of the circles be commensurable, *i.e.*, if b equals a fractional part of a, then the hypocycloid will be a closed curve; but if these radii are incommensurable, then the curve will not again pass through the initial point A.

In particular, if $a : b = 4 : 1$, then the circumference of the fixed circle is 4 times that of the rolling circle, and the hypocycloid becomes a closed curve of four arches, as shown in Fig. 134. In this case, equations (2) and (3) become, respectively,

$$\left.\begin{aligned} x &= \tfrac{3}{4}a\cos\theta + \tfrac{1}{4}a\cos 3\theta, \\ \text{and } y &= \tfrac{3}{4}a\sin\theta - \tfrac{1}{4}a\sin 3\theta. \end{aligned}\right\} \quad (4)$$

But, by trigonometry,

$$3\cos\theta + \cos 3\theta = 4\cos^3\theta,$$

and $3\sin\theta - \sin 3\theta = 4\sin^3\theta$,

hence equations (4) become

$$\left.\begin{aligned} x &= a\cos^3\theta, \\ \text{and}\qquad y &= a\sin^3\theta; \end{aligned}\right\} \quad . \quad . \quad . \quad (5)$$

whence $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ * . . . (6)

which is the common form of the equation of the four-cusped hypocycloid.

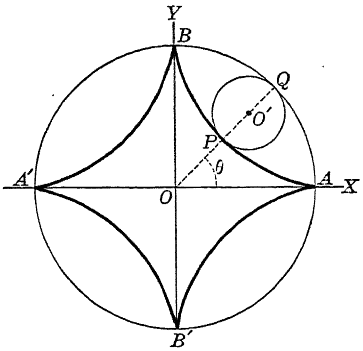


FIG. 134.

SPIRALS

193. A **spiral** is a transcendental curve traced by a point which, while it revolves about a fixed point called the **center,** also continually recedes from this center, according to some definite law.

The portion of the spiral generated during one revolution of the tracing point is called a **spire**; and the circle whose radius is the radius vector of the generating point at the end of the first revolution is called the **measuring circle** of the spiral. Thus, in Fig. 135, *ABCDE* is the measuring circle, *OQSUWA* is the first spire, and *AFHLN* is the second spire.

194. The spiral of Archimedes.† This curve is traced by a point which moves about a fixed point in a plane in such a

---

* If this equation be rationalized, it becomes

$$27\,a^2x^2y^2 = (a^2 - x^2 - y^2)^3.$$

Although the hypocycloid is, in general, a transcendental curve, it becomes algebraic for particular values of the ratio of the radii of the circles.

† This curve is usually supposed to have been discovered by Conan, though its principal properties were investigated by the geometer whose name it bears.

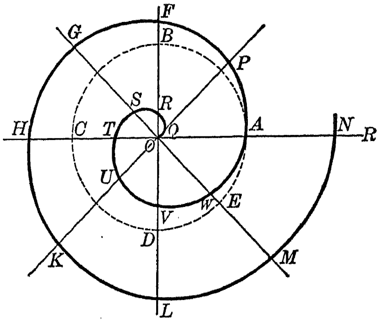


FIG. 135.

way that any two radii vectores are in the same ratio as are the angles they make with the initial line.*

From this definition it follows that the equation of the curve is

$$\rho = k\theta, \quad . \quad . \quad . \quad (1)$$

where k is a constant.

This equation shows that the locus passes through the origin, and that the radius vector becomes larger and larger without limit as the number of revolutions increases without limit. Moreover, if (ρ_1, θ_1) be any point on the curve, and if $(\rho_2, \theta_1 + 2\pi)$ be the corresponding point on the next spire, then

$$\rho_1 = k\theta_1 \text{ and } \rho_2 = k(\theta_1 + 2\pi),$$

whence

$$\rho_2 = \rho_1 + 2k\pi;$$

but $2k\pi = OA$, hence the distance between the successive points in which any radius vector meets the curve is constant; it is always equal to the radius of the measuring circle. This follows also directly from the definition.

The locus of equation (1), for positive values of θ is represented in Fig. 135; for negative values of θ the locus is symmetrical with the part already drawn, the axis of symmetry being the line LF.

195. The reciprocal or hyperbolic spiral. This curve is traced by a point which moves about a fixed point in a plane in such a way that any two radii vectores are in the same ratio as the reciprocals of the angles which they form with the initial line.

---

* This curve may also be defined thus: It is the path traced by a point which moves away from the center with uniform linear velocity, while its radius vector revolves about the center with uniform angular velocity.

From this definition it follows that the equation of the curve is

$$\rho = \frac{k}{\theta}, \quad . \quad . \quad . \quad . \qquad (1)$$

where k is a constant.

This equation shows that the curve begins at infinity when $\theta = 0$ and winds round and round the center, always approaching it, but never quite reaching it; *i.e.*, $\rho = 0$ only after an infinite number of spires have been described.

Equation (1) also shows that the constant k is the circumference of the measuring circle. For the radius of the measuring circle (Art. 193) is the radius vector of the generating point of the curve at the end of the first revolution, *i.e.*, when $\theta = 2\pi$; but, from equation (1), this radius vector is $\frac{k}{2\pi}$, and the circumference of the circle of which this is the radius is k.

Again, if $P \equiv (\rho, \theta)$ be any point on the locus of equation (1), then

$$\begin{aligned}\rho\theta &= k \\ &= \text{circumference of measuring circle};\end{aligned}$$

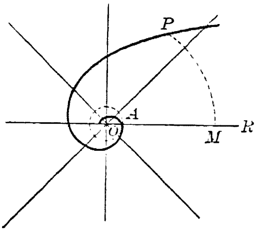


FIG. 136

but $\rho\theta$ equals the length of the circular arc described with radius ρ and subtending an angle θ, therefore the length of any circular arc as MP, described about O, with radius ρ, and extending from the initial line to the curve, is equal to the circumference of the measuring circle.

The locus of equation (1), for positive values of θ, is represented in Fig. 136.

196. The parabolic spiral. This curve is traced by a point which moves around a fixed point in a plane in such a way that the squares of any two radii vectores are in the same ratio as are the angles which they form with the initial line.

From this definition it follows that the equation of the curve is

$$\rho^2 = k\theta, \quad . \quad . \quad . \qquad (1)$$

where k is a constant.

This equation shows that the curve begins at the center when $\theta = 0$, winds round and round this point, always receding from it, the radius vector becoming infinite when θ becomes infinite, *i.e.*, when it has described an infinite number of spires.

The locus of equation (1), for positive values of ρ, is represented in Fig. 137.*

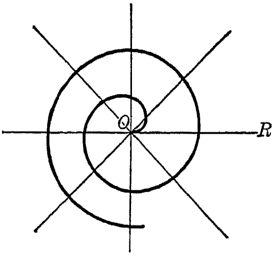


FIG. 137

197. The lituus † or trumpet. This curve is traced by a point which moves around a fixed point in a plane in such a way that the squares of any two radii vectores are in the same ratio as the reciprocals of the angles which they form with the initial line.

From this definition it follows that the equation of the curve is

$$\rho^2 = \frac{k}{\theta}, \quad . \quad . \quad . \qquad (1)$$

where k is a constant.

This equation shows that the curve begins at infinity, when $\theta = 0$, and winds round and round the center, always approaching it, but never quite reaching it, *i.e.*, $\rho = 0$ only after an infinite number of spires have been described.

---

* See also Rice and Johnson's Differential Calculus, p. 307.

† This curve was invented and named by Cotes, who died in 1716.

The locus of equation (1) is shown in Fig. 138; the heavy line being the part of the locus obtained from the positive values of ρ, while the dotted part belongs to the negative values of ρ.

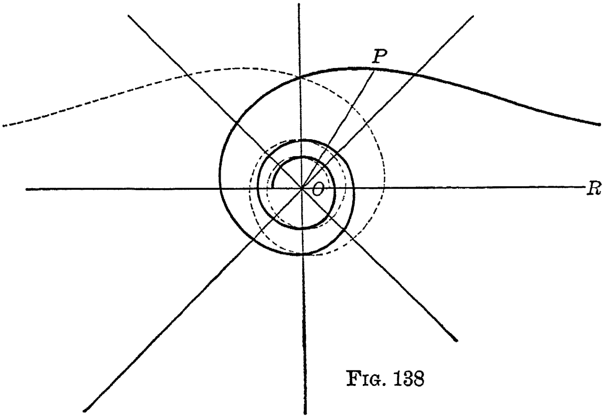


Fig. 138

Note. The four spirals just discussed, and whose forms are given in Figs. 135 to 138, are all included under the more general case of the curve defined by the equation

$$\rho = k\theta^n; \quad . \quad . \quad . \quad (2)$$

if $n = 1$, this is the spiral of Archimedes; if $n = -1$, it is the hyperbolic spiral; if $n = \frac{1}{2}$, it is the parabolic spiral; while if $n = -\frac{1}{2}$, it is the lituus.

198. The logarithmic spiral.* This curve is traced by a point which moves around a fixed point in a plane in such a way that the logarithms of any two radii vectores are in the same ratio as are the angles which these lines form with the initial line.

---

* This curve might have been defined by saying that the radius vector increases in a *geometric* ratio while the vectorial angle increases in an *arithmetic* ratio. An important property of this curve is (see McMahon and Snyder's Differential Calculus, Art. 120) that it cuts all the radii vectores at the same angle, and the tangent of this angle is the modulus of the system of logarithms which the particular spiral represents.

From this definition it follows that the equation of the curve is

$$\log \rho = k\theta, \quad . \quad . \quad . \quad (1)$$

where k is a constant.

If k be unity, and logarithms to the base a be employed, this equation may be written in the form

$$\rho = a^{\theta}. \quad . \quad . \quad . \quad (2)$$

This equation shows that if $\theta = -\infty$, $\rho = 0$; that ρ increases from 0 to 1, while θ increases from $-\infty$ to 0; and that ρ continues to increase from 1 to ∞, while θ increases from 0 to $+\infty$; the curve has, therefore, an infinite number of spires.

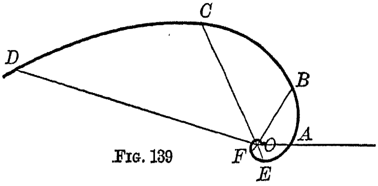


FIG. 139

If the constant a equals 2, then ρ takes the values $\cdots \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, \cdots$, when θ is assigned the values (in radians), $\cdots, -2, -1, 0, 1, 2, 3, \cdots$; Fig. 139 represents the locus of equation (2), a being equal to 2, for values of θ from -2π to $+3$. In this figure $\angle FOE = \angle EOA = \angle AOB = \angle BOC = \angle COD = 57°.3$, and $OF = \frac{1}{4}$, $OE = \frac{1}{2}$, $OA = 1$, $OB = 2$, $OC = 4$, and $OD = 8$.

PART II

SOLID ANALYTIC GEOMETRY

CHAPTER I

COÖRDINATE SYSTEMS. THE POINT

199. Solid Analytic Geometry treats by analytic methods problems which concern figures in space, and therefore involve three dimensions. It is evident that new systems of coördinates must be chosen, involving three variables; and that the analytic work will therefore be somewhat longer than in the plane geometry. On the other hand, since a plane may be considered as a special case of a solid where one dimension has the particular value zero, it is to be expected that the analytic work with three coördinate variables should be entirely consistent with that for two variables; merely a simple extension of the latter. The student should not fail to notice this close analogy in all cases.

In the present chapter will be considered some simple and useful systems of coördinates for determining the position of a point in space, some elementary problems concerning points, and the transformations of coördinates from one system to another. Later chapters will treat briefly of surfaces, particularly of planes and of surfaces of the second order, and of the straight line.

200. Rectangular coördinates. Let three planes be given fixed in space and perpendicular to each other, — the **coördinate planes** XOY, YOZ, and ZOX. They will intersect by pairs in three lines, $X'X$, $Y'Y$, and $Z'Z$, also perpendicular to each other, called the **coördinate axes.** And these three lines will meet in a common point O, called the **origin.** Any three other planes, MP, NP, and LP, parallel respectively to these coördinate planes, will intersect in three lines, $N'P$, $L'P$, $M'P$, which will be parallel respectively to the axes; and these three lines will meet in, and completely determine, a point P in space. The directed distances $N'P$, $L'P$, and $M'P$ thus determined, *i.e.*, the perpendicular distances of the point P from the coördinate planes, are the **rectangular coördinates** of the point P. They are represented respectively by x, y, and z. It is clear that

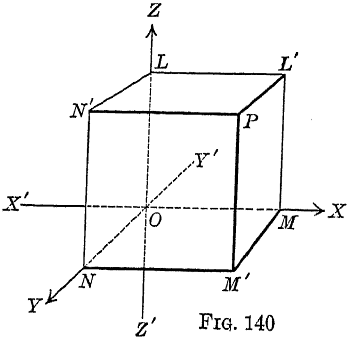


FIG. 140

$$x = N'P = LL' = NM' = OM;$$
$$y = L'P = MM' = LN' = ON;$$
$$z = M'P = NN' = ML' = OL.$$

It is generally convenient, however, to consider

$$x = OM,\ y = MM',\ \text{and}\ z = M'P.$$

The point may be denoted by the symbol $P \equiv (x, y, z)$.

The axes may be directed at pleasure; it is usual to take the positive directions as shown in the figure. Then the eight portions, or octants, into which space is divided by the coördinate planes, will be distinguished completely by the signs of the coördinates of points within them.

If the chosen coördinate planes were oblique to each other, a set of oblique coördinates for any point in space might be found in an entirely analogous way.

Unless otherwise stated, rectangular coördinates will be used in the subsequent work.

201. Polar coördinates. A second method of fixing the position of a point in space is by means of its distance and direction from a given fixed point. Let O be a fixed point in space, called the **pole**; and let ρ be the distance from O to any other point P. To give the direction of ρ, let OR and OS be two chosen directed perpendicular lines through O, determining the plane ROS; then the direction of ρ will be given by the angle θ from the plane ROS to the plane POM, and the angle ϕ from the line OS to ρ. The point P is completely determined by the values of its **radius vector** ρ and its **vectorial angles** θ and ϕ, and may be denoted as $P \equiv (\rho, \theta, \phi)$. The elements ρ, θ, ϕ are called the **polar coordinates** of the point P.

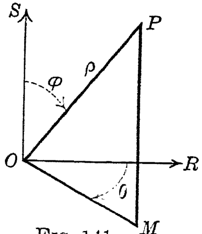


FIG. 141

It is to be noted that for convenience the positive values of θ and ϕ are those for rotation in *clockwise* direction from ROS and OS, respectively. And although a given set of coördinates fixes a single point, yet any point may have eight sets of coördinates in a polar system, if, as usual, the values of the angles are less than 360°.

202. Relation between the rectangular and polar systems. If the axes OR and OS of a polar system coincide with the axes OX and OZ, respectively, of a rectangular sys-

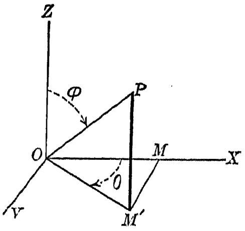


Fig. 142

tem, the pole and origin therefore being coincident, then simple relations exist between the two sets of coördinates for any point. For, since $\angle OMM' = 90°$ and $\angle OM'P = 90°$, therefore

$$OM = OM' \cos\theta$$

$$= OP \sin\phi \cos\theta.$$

$$MM' = OM' \sin\theta = OP \sin\phi \sin\theta,$$

and $$M'P = OP \cos\phi;$$

that is,

$$\left.\begin{aligned} x &= \rho\cos\theta\sin\phi, \\ y &= \rho\sin\theta\sin\phi, \\ z &= \rho\cos\phi. \end{aligned}\right\} \quad . \quad . \quad . \quad [1]$$

Again, $$\overline{OP}^2 = \overline{OM'}^2 + \overline{M'P}^2 = \overline{OM}^2 + \overline{MM'}^2 + \overline{M'P}^2.$$

$$\left.\begin{aligned} &\textit{i.e.,} && \rho^2 = x^2 + y^2 + z^2, \\ &\text{also} && \tan\theta = \frac{y}{x}, \\ &\text{and} && \cos\phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}}. \end{aligned}\right\} \quad . \quad . \quad . \quad [2]$$

The above relations give formulas for transformation from the one coördinate system to the other.

203. Direction angles: direction cosines. A third useful method of fixing a point in space is a combination of the two methods already considered. The axes of reference are chosen as in rectangular coördinates, and any point P of space is fixed by its distance from the origin, called the **radius vector,** and the angles α, β, γ, which this radius

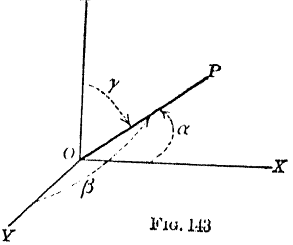


Fig. 143

vector makes with the coördinate axes, respectively. These angles are called the **direction angles** of the line OP, and their cosines, its **direction cosines.** The point may be concisely denoted as the point $P \equiv (\rho, \alpha, \beta, \gamma)$.

Simple equations connect these coördinates with those of the rectangular system; for, projecting OP upon the axes OX, OY, and OZ, respectively,

$$x = \rho \cos \alpha, \quad y = \rho \cos \beta, \quad z = \rho \cos \gamma, \quad . \quad . \quad . \quad [3]$$

and also, $\rho^2 = x^2 + y^2 + z^2$, as in equations [2].

Moreover, the direction cosines are not independent, but are connected by an equation; for, by combining the above equations,

$$\rho^2 = \rho^2 \cos^2 \alpha + \rho^2 \cos^2 \beta + \rho^2 \cos^2 \gamma,$$

i.e.,

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1. \quad . \quad . \quad . \quad [4]$$

Such a relation was to have been expected, since only three magnitudes are necessary to determine the position of a point, and therefore the four numbers ρ, α, β, γ could not be independent.

Any three numbers, a, b, c, are proportional to the direction cosines of some line; because if these numbers are considered as the coördinates of a point, then the direction cosines of the radius vector of that point are, by eq. [3],

$$\cos \alpha = \frac{a}{\sqrt{a^2+b^2+c^2}}, \cos \beta = \frac{b}{\sqrt{a^2+b^2+c^2}}, \cos \gamma = \frac{c}{\sqrt{a^2+b^2+c^2}}. \quad [5]$$

These direction cosines are proportional to a, b, c; and are found by dividing a, b, c, respectively, by the same constant,

$$\sqrt{a^2+b^2+c^2}.$$

Direction cosines are useful in giving the direction of any line in space. The direction of any line is the same as that of a parallel line through the origin, therefore the direction of a line may be given by the direction angles of some point whose radius vector is parallel to the line. Sometimes, as an equivalent conception, it is convenient to consider the direction angles as those formed by the line with three lines which pass through some point of the given line, and are parallel, respectively, to the coördinate axes.

204. Distance and direction from one point to another; rectangular coördinates. A few elementary problems concerning points can now be easily solved; for example, the problem of finding the distance between two points. Let OX, OY, OZ be a set of rectangular axes, and $P_1 \equiv (x_1, y_1, z_1)$ and $P_2 \equiv (x_2, y_2, z_2)$ be two given points. Then the planes through P_1 and P_2, parallel, respectively, to the coördinate planes, form a rectangular parallelopiped, of which the required distance P_1P_2 is a diagonal. From the figure,

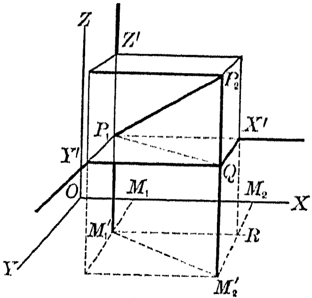


Fig. 144

since $\angle P_1QP_2 = 90°$ and $\angle M_1'RM_2' = 90°$,

therefore
$$\overline{P_1P_2}^2 = \overline{P_2Q}^2 + \overline{QP_1}^2 = \overline{M_1'M_2'}^2 + \overline{QP_1}^2$$
$$= \overline{M_2'R}^2 + \overline{RM_1'}^2 + \overline{QP_1}^2$$
$$= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.$$

That is, if d be the required distance,

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \quad . \; . \; . \quad [6]$$

Moreover, since the direction of the line P_1P_2 is given by the angles α, β, γ, which it makes, respectively, with the lines P_1X', P_1Y', and P_1Z', drawn through P_1 parallel to the axes; then projection of $d = P_1P_2$ upon these lines in turn gives

$$P_1P_2 \cos\alpha = P_1X', \quad P_1P_2 \cos\beta = P_1Y', \quad P_1P_2 \cos\gamma = P_1Z',$$

i.e., $d\cos\alpha = x_2 - x_1, \quad d\cos\beta = y_2 - y_1, \quad d\cos\gamma = z_2 - z_1,$

and, finally,

$$\cos\alpha = \frac{x_2 - x_1}{d}, \quad \cos\beta = \frac{y_2 - y_1}{d}, \quad \cos\gamma = \frac{z_2 - z_1}{d}. \quad . \quad . \quad . \quad [7]$$

These equations give the required direction angles of P_1P_2.

205. The point which divides in a given ratio the straight line from one point to another. Let $P_1 \equiv (x_1, y_1, z_1)$ and $P_2 \equiv (x_2, y_2, z_2)$ be two given points, and let $P_3 \equiv (x_3, y_3, z_3)$ be a third point which divides the line P_1P_2 in the given ratio $\frac{m_1}{m_2}$, so that $\frac{P_1P_3}{P_3P_2} = \frac{m_1}{m_2}$.

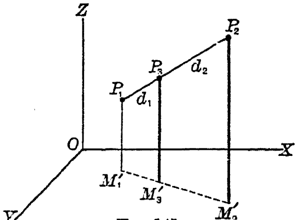


FIG. 145

Let $P_1P_3 = d_1$, and $P_3P_2 = d_2$; then by Art. 204, if α, β, γ be the direction angles of P_1P_2,

$$\cos\alpha = \frac{x_3 - x_1}{d_1} = \frac{x_2 - x_3}{d_2}; \quad \therefore \frac{x_3 - x_1}{x_2 - x_3} = \frac{d_1}{d_2} = \frac{m_1}{m_2},$$

$$\left.\begin{aligned} &\text{and} && x_3 = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}. \\ &\text{Similarly,} && y_3 = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}, \\ &\text{and} && z_3 = \frac{m_1z_2 + m_2z_1}{m_1 + m_2}. \end{aligned}\right\} \quad . \quad . \quad . \quad [8]$$

It will be noticed, as in the similar problem in Part I, Art. 30, that if P_3 divides the line externally, the ratio $\frac{m_1}{m_2}$ must be negative; and the above formulas still apply.

If P_3 bisects the line P_1P_2, formulas [8] take the simpler forms

$$x_3 = \frac{x_1 + x_2}{2}, \quad y_3 = \frac{y_1 + y_2}{2}, \quad z_3 = \frac{z_1 + z_2}{2}. \quad . \quad . \quad [9]$$

206. Angle between two radii vectores. Angle between two lines. Let $P_1 \equiv (\rho_1, \alpha_1, \beta_1, \gamma_1)$ and $P_2 \equiv (\rho_2, \alpha_2, \beta_2, \gamma_2)$ be two given points, and θ the angle included by the radii vectores ρ_1 and ρ_2. Then the projections upon OP_1 of the line OP_2 and of the broken line $OM_2M_2'P_2$ are equal (Art. 17); hence,

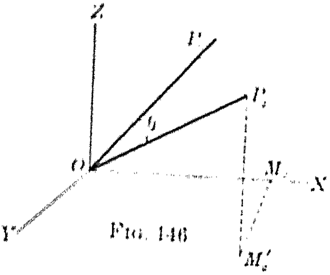


FIG. 146

$$\text{proj. } OP_2 = \text{proj. } OM_2M_2'P_2,$$

i.e., $\rho_2 \cos\theta = OM_2 \cos\alpha_1 + M_2M_2' \cos\beta_1 + M_2'P_2 \cos\gamma_1$.

But $$OM_2 = \rho_2 \cos\alpha_2,$$

$$M_2M_2' = \rho_2 \cos\gamma_2, \text{ and } M_2'P_2 = \rho_2 \cos\gamma_2;$$

hence,

$$\rho_2 \cos\theta = \rho_2 \cos\alpha_2 \cos\alpha_1 + \rho_2 \cos\beta_2 \cos\beta_1 + \rho_2 \cos\gamma_2 \cos\gamma_1;$$

i.e., $$\cos\theta = \cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2, \qquad [10]$$

and this relation determines the required angle θ.

It follows, since any two straight lines in space have their directions given by the direction angles of radii vectores which are parallel to them, respectively, that formula [10] applies as well to the angle θ between any two straight lines in space, whose direction angles are given.

Two special cases arise, of parallel and of perpendicular lines. If the two given lines are parallel, evidently

$$\alpha_1 = \alpha_2, \ \beta_1 = \beta_2, \ \gamma_1 = \gamma_2; \qquad [11]$$

and formula [10] reduces to eq. [4]. If the lines are perpendicular, $\cos\theta = 0$, and eq. [10] reduces to

$$\cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2 = 0. \quad . \quad . \quad . \quad [12]$$

207. Transformation of coördinates; rectangular systems. The relations found in Art. 202 to exist between rectangular and polar coördinates of a point may be used as formulas of transformation from one system to the other if the origin, the pole, and the reference axes are coincident. Two other simple transformations may be useful, (1) from one set of rectangular coördinates to a parallel set, *i.e.*, a change of *origin* only; and (2) from one set of rectangular axes to another set through the same origin, *i.e.*, a change of *direction* of axes. Then any transformation between rectangular and polar systems can be performed by a combination of these three elementary transformations.

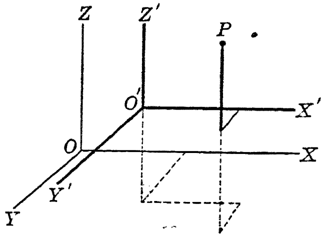


FIG. 147

(1) *Change of origin only.* Let the new origin be the point $O' \equiv (h, k, j)$; then, constructing the coördinates of any point P with reference to each set of coördinate planes, it is evident, by analogy with Art. 71, that

$$x = x' + h, \quad y = y' + k, \quad z = z' + j. \quad . \quad . \quad . \quad [13]$$

(2) *Change of direction of axes.* Let a second set of rectangular axes, OX', OY', OZ', have the direction angles α_1, β_1, γ_1, α_2, β_2, γ_2, and α_3, β_3, γ_3, respectively, with the old axes OX, OY, OZ.

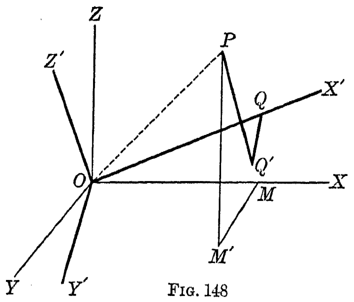


FIG. 148

Then if the coördinates of any point P in the two systems are

$$x = OM,$$
$$y = MM',$$
$$z = M'P,$$

and

$$x' = OQ,$$
$$y' = QQ',$$
$$z' = Q'P,$$

then projections of OP and the broken line $OQQ'P$ upon OX, OY, OZ, in turn, will be equal; hence,

$$\left.\begin{aligned} x &= x' \cos \alpha_1 + y' \cos \alpha_2 + z' \cos \alpha_3, \\ y &= x' \cos \beta_1 + y' \cos \beta_2 + z' \cos \beta_3, \\ z &= x' \cos \gamma_1 + y' \cos \gamma_2 + z' \cos \gamma_3. \end{aligned}\right\} \quad . \; . \; . \quad [14]$$

These formulas are for transformation from the first system to the second. But, also, by projecting OP and $OMM'P$ upon OX', OY', OZ', respectively,

$$\left.\begin{aligned} x' &= x \cos \alpha_1 + y \cos \beta_1 + z \cos \gamma_1, \\ y' &= x \cos \alpha_1 + y \cos \beta_2 + z \cos \gamma_2, \\ z' &= x \cos \alpha_3 + y \cos \beta_3 + z \cos \gamma_3, \end{aligned}\right\} \quad . \; . \; . \quad [15]$$

and these formulas are for the reverse transformation, from the second system to the first.

NOTE. It is to be remembered that in the transformation of [14] and [15], twelve conditions exist, by eq. [4] and eq. [12], three of each of the following types,

$$\cos^2 \alpha_1 + \cos^2 \alpha_2 + \cos^2 \alpha_3 = 2,$$
$$\cos^2 \alpha_1 + \cos^2 \beta_1 + \cos^2 \gamma_1 = 1,$$
$$\cos \alpha_1 \cos \alpha_2 + \cos \beta_1 \cos \beta_2 + \cos \gamma_1 \cos \gamma_2 = 0,$$
$$\cos \alpha_1 \cos \beta_1 + \cos \alpha_2 \cos \beta_2 + \cos \alpha_3 \cos \beta_3 = 0.$$

These equations are not independent, however, but reduce to six independent equations.

It is clear, by reasoning similar to that of Art. 75, Part I, that none of the transformations [13], [14], and [15], neither separately nor in combination, can alter the degree of an equation to which they may be applied.

EXAMPLES ON CHAPTER I

1. Prove that the triangle formed by joining the points (1, 2, 3), (2, 3, 1), and (3, 1, 2), in pairs, is equilateral.

2. The direction cosines of a straight line are proportional to 1, 2, 3; find their values.

3. Find the angle between two straight lines whose direction cosines are proportional to 2, 2, 2, and 5, -4, 7, respectively.

4. The rectangular coördinates of a point are $(\sqrt{3}, 1, 2\sqrt{3})$; find its polar coördinates.

5. The polar coördinates of a point are $\left(3, \frac{\pi}{6}, \frac{\pi}{4}\right)$; find its rectangular coördinates.

6. Express the distance between two points in terms of their polar coördinates.

7. Find the coördinates of the points dividing the line from $(-2, -3, 1)$ to $(3, -2, 4)$ externally and internally in the ratio $2:5$.

8. What is the length of a line whose projections on the coördinate axes are 4, 1, 3, respectively?

9. Find the radius vector, and its direction cosines, for each of the points $(-7, 1, 5)$, $(1, -1, -2)$, $(a, 0, b)$.

10. Find the center of gravity* of the triangle of Ex. 1.

11. Find the direction angles of a straight line which makes equal angles with the three coördinate axes.

12. A straight line makes the angle 30° with the x-axis, and 75° with the z-axis. At what angle does it meet the y-axis?

13. Prove analytically that the straight lines joining the mid-points of the opposite edges of a tetrahedron pass through a common point, and are bisected by it.

14. Prove analytically that the straight lines joining the mid-points of the opposite sides of any quadrilateral pass through a common point, and are bisected by it.

---

* See Ex. 15, p. 42.

CHAPTER II

THE LOCUS OF AN EQUATION. SURFACES

208. Attention has been called to the close analogy between the corresponding analytical results for the geometry of the plane and of space. It is evident that in geometry of one dimension, restricted to a line, the point is the elementary conception. Position is given by one variable, referring to a fixed point in that line; and any algebraic equation in that variable represents one or more points. In geometry of two dimensions, however, it has been shown that the line may be taken as the fundamental element. Position is given by two variables, referring to two fixed lines * in the plane; and any algebraic equation in the two variables represents a curve, *i.e.*, a line whose generating point moves so as to satisfy some condition or law. Correspondingly, in geometry of three dimensions the surface is the elementary conception. Position is given by three variables, referring to three fixed surfaces, since any point is the intersection of three surfaces; † and it can be shown that any algebraic equation in three variables represents some surface.

---

* With polar coördinates, these lines are a circle about the pole with radius $= \rho$, and a straight line through the pole making the angle θ with the initial line (Art. 23).

† With polar coördinates, these surfaces are a sphere, about the origin as center, determined by the radius vector ρ, a right cone about the z-axis, with vertex at the origin, determined by the angle ϕ, and a plane through the z-axis determined by the angle θ (Art. 201).

The study of the special equations of first and second degree will be taken up in the two succeeding chapters. Here it is desired to show that an algebraic equation in three variables represents a surface, and to consider briefly two simple classes of surfaces: (1) **cylinders,** *i.e.*, surfaces which are generated by a straight line moving parallel to a fixed straight line, and always intersecting a fixed curve; and (2) **surfaces of revolution,** *i.e.*, surfaces generated by revolving some plane curve about a fixed straight line lying in its plane.

209. Equations in one variable. Planes parallel to coördinate planes. From the definition of rectangular coördinates, it follows that the equations

$$x = 0,\ \ y = 0,\ \ z = 0,$$

represent the coördinate planes, respectively, and that any algebraic equation in one variable and of the first degree represents a plane parallel to one of them. Similarly, an equation in one variable and of degree n will represent n such parallel planes, either real or imaginary. For, any such equation, as

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0, \quad . \quad . \quad (1)$$

can be factored into n linear factors, real or imaginary,

$$p_0(x - x_1)(x - x_2)(\cdots)(x - x_n) = 0; \quad . \quad . \quad . \quad (2)$$

and by the reasoning of Part I, Art. 40, eq. (2) will represent the loci of the n equations

$$x - x_1 = 0,\ \ x - x_2 = 0,\ \ \cdots,\ \ x - x_n = 0,$$

each of which is a plane, parallel to the yz-plane, and real if the corresponding root is real. In the same way, an equation in y or z only will represent planes parallel to the zx- or xy-plane.

Any algebraic equation in one variable represents one or more planes parallel to a coördinate plane.

It follows at once by Art. 39, that two simultaneous equations of the first degree in one variable represent the intersection of two planes parallel to coördinate planes; therefore, represent a straight line parallel to the coördinate axis of the third variable; *e.g.*, $y = b$, $z = c$, considered as simultaneous equations, represent a straight line parallel to the x-axis.

210. Equations in two variables. Cylinders perpendicular to coördinate planes.

Consider the equation

$$2x + 3y = 6, \quad . \quad . \quad . \quad (1)$$

with two variables only. In the xy-plane it represents a straight line AB. If, now, from any point P of AB a straight line be drawn parallel to the z-axis, the x and y coördinates of every point Q on this line will be the same as for P, and therefore satisfy equation (1). Moreover, if the line PQ moved along AB, and always parallel to the z-axis,

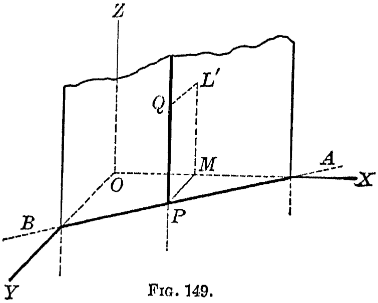


Fig. 149.

still the coördinates of every point in it satisfy equation (1). As the line PQ is thus moved, it traces a plane surface perpendicular to the xy-plane; and, as evidently the coördinates of a point not on this surface do not satisfy equation (1) this cylindrical plane is the locus of equation (1).

Again: the equation

$$y^2 + z^2 = r^2 \quad . \quad . \quad . \qquad (2)$$

represents in the yz-plane a circle. It is therefore satisfied by the coördinates of any point Q, in a line parallel to the x-axis, through any point P of this circle; and also by the coördinates of Q as this line PQ is moved, parallel to the x-axis and along the circle. The circular cylinder thus traced by the line PQ, perpendicular to the yz-plane, is the locus of the given equation.

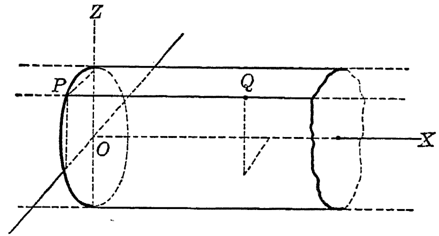


FIG. 150.

Similarly, it may be shown that the locus of the equation

$$\frac{x^2}{a^2} - \frac{z^2}{b^2} = 1 \quad . \quad . \quad . \qquad (3)$$

is a cylindrical surface traced by a straight line parallel to the y-axis, and moving along the hyperbola whose equation in the xz-plane is equation (3). And, in general, it is clear by analogy that *any algebraic equation in two variables represents a cylindrical surface whose elements are parallel to the* *axis of the third variable, and having its form and position determined by the plane curve represented by the same equation.*

As a direct consequence, it is clear that if a cylinder has its axis parallel to a coördinate axis, a section made by a plane, perpendicular to that axis, is a curve parallel to and equal to the directing curve on the coördinate plane, and is represented in the cutting plane by the same equation. Thus, the section of the elliptical cylinder whose equation is $3x^2 + y^2 = 5$, cut by the plane $z = 7$, is an ellipse equal and parallel to the ellipse whose equation is $3x^2 + y^2 = 5$.

211. Equations in three variables. Surfaces. A solid figure has the distinctive property that it can be cut by a straight line in an infinite number of points, while a surface or line can, in general, be cut in only a finite number. A line has the distinctive property that it can be, in general, cut by a plane in only one point, while a surface may be cut in a curve. To show that the locus of an algebraic equation in three variables is, in general, a surface, it is sufficient to show that, in general, a plane will cut it in a curve, while a straight line will cut it in a finite number of points.

Let the given equation be

$$f(x, y, z) = 0, \quad . \quad . \quad . \quad (1)$$

and let

$$z = c \quad . \quad . \quad . \quad (2)$$

be a plane parallel to the xy-plane. The points of intersection of these two loci will be on the locus of the equation

$$f(x, y, c) = 0; \quad . \quad . \quad . \quad (3)$$

and, by Art. 210, they lie, therefore, upon a plane curve, cut from the cylinder whose equation is (3), by the plane whose equation is (2). Hence the locus of equation (1) is not a line.

Again, let $y = b, \quad z = c \quad . \quad . \quad . \qquad (4)$

be the equations of a straight line (Art. 209), parallel to the x-axis. The points of intersection of locus (1) and the line (4) will be also on the locus of the equation

$$f(x, b, c) = 0 \; ; \quad . \quad . \quad . \qquad (5)$$

which, since the equation is in one variable, of finite degree, will represent a finite number of planes parallel to the yz-plane, and therefore having a finite number of points of intersection with the line (4). Hence the locus of equation (1) is not a solid.

Therefore, *the locus of any algebraic equation in three variables is a surface.*

212. Curves. Traces of surfaces. Two surfaces intersect in a curve in space; and since every algebraic equation in solid analytic geometry represents a surface, a curve may be represented analytically by the two equations, regarded as simultaneous, of surfaces which pass through it. Thus it has been seen that the equations $y = b$, $z = c$ separately represent planes, but considered as simultaneous represent the straight line which is the intersection of those planes. But by the reasoning of Art. 41, the given equations of a curve may be replaced by simpler ones which represent other surfaces passing through the same curve. In dealing with curves it is often useful to obtain, from the equations given, equations of cylinders through the same curve; *i.e.*, it is generally useful to represent a curve by two equations each in two variables only.

EXAMPLE: The curve of intersection of the two surfaces,

$$(1) \; x^2 + y^2 + z^2 - 25 = 0 \quad \text{and} \quad (2) \; x^2 + y^2 - 16 = 0,$$

is also the intersection of the surfaces

$$x^2 + y^2 + z^2 - 25 - (x^2 + y^2 - 16) = 0, \text{ i.e., } z = \pm 3, \qquad (3)$$

with the surface (2). The curve is therefore composed of two circles of radius 4, parallel to the xy-plane at distances $+3$ and -3 from it.

Conversely, the curves of intersection of a surface with the coördinate planes may be used to help determine the nature of a surface. These curves are called the **traces** of the surface.

Thus, the surface $x^2 + y^2 + z^2 = 25$ has the traces

on the yz-plane, where $x = 0$, $y^2 + z^2 = 25$;
on the zx-plane, where $y = 0$, $x^2 + z^2 = 25$;
on the xy-plane, where $z = 0$, $x^2 + y^2 = 25$.

Each of these traces is a circle of radius 5, about the origin as center; the surface is a sphere of radius 5 with center at the origin.

Since three surfaces in general have only one or more separate points in common, the locus of three equations, considered as simultaneous, is one or more distinct points.

213. Surfaces of revolution. Analogous to the cylinders are the surfaces traced by revolving any plane curve about a straight line in the plane as axis. From the method of formation, it follows that each plane section perpendicular to the axis is a circle, — the path traced by a point of the generating curve as it revolves; and the radius of the circle is the distance of the point from the axis in the plane before revolution begins. These facts lead readily to the equation of any surface of revolution, as a few examples will show.

(*a*) *The cone formed by revolving about the z-axis the line*

$$2x + 3z = 15. \quad . \quad . \quad . \quad (1)$$

Any point P of the line (1) traces during the revolution a circle of radius LP, parallel to the xy-plane. The equation of that path is

$$x^2 + y^2 = \overline{LP}^2.$$

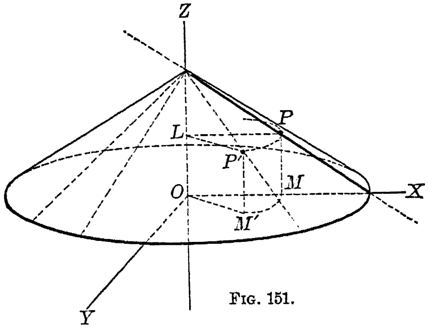


FIG. 151.

But in the xz-plane, before revolution is begun, LP is the abscissa x of P; hence, by equation (1),

$$\overline{LP} = x = \frac{15 - 3z}{2};$$

so that the equation of the path of P is

$$x^2 + y^2 = \frac{(15 - 3z)^2}{4}. \quad . \quad . \quad . \quad (2)$$

But P is *any* point of line (1); hence equation (2) is satisfied by every point of the line, and represents the surface generated by the line, which is the required conical surface.

(*b*) *The sphere formed by revolving about the* y*-axis the circle*

$$x^2 + y^2 = 25. \quad . \quad . \quad . \quad (3)$$

In this case, any point P of the curve traces during the revolution a circle of radius NP, parallel to the zx-plane. The equation of this path is therefore

$$x^2 + z^2 = \overline{NP}^2.$$

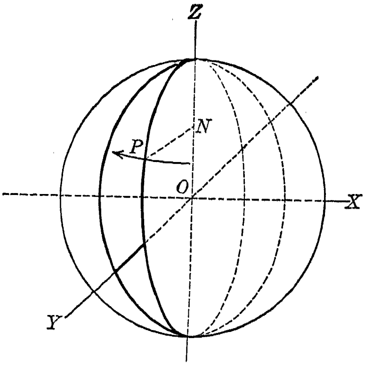


Fig. 152.

But in the xy-plane, by equation (3)

$$NP = x = \sqrt{25 - y^2}.$$

Hence, substituting above,

$$x^2 + z^2 = 25 - y^2,$$

i.e., $$x^2 + y^2 + z^2 = 25; \quad (4)$$

which is the equation of the required spherical surface.

(*c*) *The surface formed by revolving about the x-axis the curve*

$$z = (x-1)(x-2)(x-3) \text{ [cf. Art. 37]}. \quad . \quad . \quad . \quad (5)$$

Any point P of the generating curve traces a circle parallel to the yz-plane, with a radius MP equal to the z-abscissa in equation (5). Hence the equation of its path is

$$y^2 + z^2 = \overline{MP}^2,$$

i.e., $$y^2 + z^2 = (x-1)^2 (x-2)^2 (x-3)^2; \ldots (6)$$

which is the equation of the required surface.

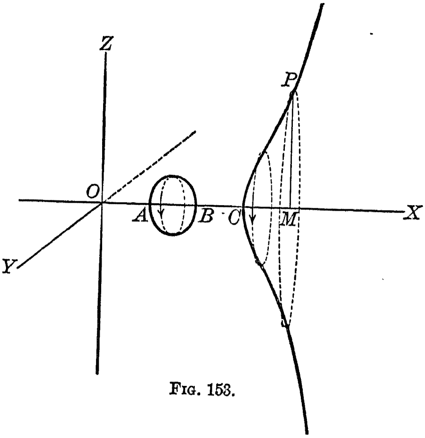


Fig. 153.

(*d*) Of the various surfaces of revolution those of particular interest are generated by revolving about their axes the various conic sections, giving the cones, spheres, paraboloids, ellipsoids, and hyperboloids of revolution.

The student may verify the equations of the following surfaces: *

The sphere: with center at the point (a, b, c), and radius r,

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2; \quad . \quad . \quad . \quad (7)$$

with center at the origin, and radius r,

$$x^2+y^2+z^2=r^2. \quad . \quad . \quad . \quad (8)$$

The cone: the surface generated by the right line $z=mx+c$, rotated about the z-axis,

$$x^2+y^2=\frac{(z-c)^2}{m^2}. \quad . \quad . \quad . \quad (9)$$

The oblate spheroid: the surface generated by the ellipse $\frac{x^2}{a^2}+\frac{z^2}{b^2}=1$, rotated about the *minor* axis,

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{b^2}=1. \quad . \quad . \quad . \quad (10)$$

The prolate spheroid: the surface generated by the ellipse $\frac{x^2}{b^2}+\frac{z^2}{a^2}=1$, rotated about the *major* axis,

$$\frac{x^2}{b^2}+\frac{y^2}{b^2}+\frac{z^2}{a^2}=1. \quad . \quad . \quad . \quad (11)$$

The hyperboloid of one nappe: the surface generated by the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, rotated about the *conjugate* axis,

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}-\frac{z^2}{b^2}=1. \quad . \quad . \quad . \quad (12)$$

The hyperboloid of two nappes: the surface generated by the hyperbola $\frac{x^2}{b^2}-\frac{y^2}{a^2}=1$, rotated about the *transverse* axis,

$$\frac{x^2}{b^2}+\frac{y^2}{b^2}-\frac{z^2}{a^2}=-1. \quad . \quad . \quad . \quad (13)$$

---

* See Chap. IV, where diagrams are given for the corresponding cases of the general quadric, with elliptical instead of circular sections.

The paraboloid of revolution: the surface generated by the parabola $x^2 = 4pz$, rotated about its axis,

$$x^2 + y^2 = 4pz. \quad . \quad . \quad . \qquad (14)$$

EXAMPLES ON CHAPTER II

What is the locus of each of the following equations?

1. $x^2 - 6x + 9 = 0$.

2. $2x + 4 = 0$.

3. $x^2 - 2xy + y^2 + 2x - 2y + 1 = 0$.

4. $ax^2 + bxy + cy^2 = 0$.

5. $4yz + 6y - 8z + 1 = 0$.

6. $z^2 - 9y = 0$.

What are the curves of intersection of the surfaces represented by the equations

7. $y + 3 = 0, \quad 3x^2 + 3y^2 + 3z^2 = 20$?

8. $x^2 - y^2 = 0, \quad z = a$?

9. $x^2 + y^2 + z^2 = 9, \quad 4x^2 + y^2 = 4$?

10. $9(x^2 + y^2) - z^2 = 25 - 10z, \quad z = \pm 5$?

11. $3x^2 - 4y^2 - z^2 = 12, \quad \dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$?

Determine the projections upon the coördinate planes of the following surfaces:

12. $x^2 + y^2 + 4z^2 = 25$;

13. $3x^2 - 4y^2 - z^2 = 12$.

Find the equation of

14. the paraboloid of revolution one of whose traces is $y^2 = -5x + 3$.

15. the curve of revolution one of whose traces is $y = -5x + 3$ and whose axis is the axis of y. Find its vertex.

16. the oblate spheroid one of whose traces is $\dfrac{z^2}{2} + \dfrac{x^2}{3} = 1$.

17. the prolate spheroid one of whose traces is $\dfrac{y^2}{7} + \dfrac{z^2}{9} = 1$.

18. the surface of revolution whose axis is the axis of x and one of whose traces is $x^2y - 1 = 0$.

19. the hyperboloid of two nappes one of whose traces is $16x^2 - 9z^2 = 1$.

20. the sphere described about the major axis of the ellipse $4x^2 + 9y^2 - 24x = 0$ as diameter.

CHAPTER III

EQUATIONS OF THE FIRST DEGREE

$Ax + By + Cz + D = 0$

PLANES AND STRAIGHT LINES

I. THE PLANE

214. Every equation of the first degree represents a plane. A plane is a surface such that it contains every point on a straight line joining any two of its points.

Let $P_1 \equiv (x_1, y_1, z_1)$ and $P_2 \equiv (x_2, y_2, z_2)$ be any two points of the surface whose equation is

$$\boldsymbol{Ax + By + Cz + D = 0}, \quad . \quad . \quad . \quad [16]$$

so that $$Ax_1 + By_1 + Cz_1 + D = 0 \quad . \quad . \quad . \quad (1)$$

and $$Ax_2 + By_2 + Cz_2 + D = 0. \quad . \quad . \quad . \quad (2)$$

Now, if $P_3 \equiv (x_3, y_3, z_3)$ be any point on the straight line from P_1 to P_2 at a distance d_1 from P_1 and d_2 from P_2, then, by Art. 205,

$$x_3 = \frac{d_1x_2 + d_2x_1}{d_1 + d_2}, \quad y_3 = \frac{d_1y_2 + d_2y_1}{d_1 + d_2}, \quad z_3 = \frac{d_1z_2 + d_2z_1}{d_1 + d_2}. \quad (3)$$

But this point lies on the surface represented by equation [16]; for, substituting its coördinates from (3) in equation [16], the latter becomes

$$\frac{d_1}{d_1 + d_2}(Ax_2 + By_2 + Cz_2 + D) + \frac{d_2}{d_1 + d_2}(Ax_1 + By_1 + Cz_1 + D) = 0,$$

which is a true equation, since each parenthesis vanishes separately by equations (1) and (2). Hence every point of the line P_1P_2 is on the locus of equation [16], and that locus is therefore a plane. *Every algebraic equation of the first degree in three variables represents a plane.*

215. Equation of a plane through three given points. The general equation of the first degree,

$$Ax + By + Cz + D = 0, \quad . \quad . \quad . \quad (1)$$

has only three arbitrary constants, viz. the ratios of the coefficients. If three given points in the plane are

$$P_1 \equiv (x_1, y_1, z_1), \quad P_2 \equiv (x_2, y_2, z_2), \quad \text{and} \quad P_3 \equiv (x_3, y_3, z_3),$$

then these ratios may be found from the three equations,

$$\left.\begin{aligned} Ax_1 + By_1 + Cz_1 + D = 0, \\ Ax_2 + By_2 + Cz_2 + D = 0, \\ Ax_3 + By_3 + Cz_3 + D = 0, \end{aligned}\right\} \quad . \quad . \quad . \quad (2)$$

considered as simultaneous.

In solving equation (2) for the required ratios, two special cases may occur: (*a*) The value of one of the coefficients may be zero, then the ratios determined must not have that coefficient in the denominator. *E.g.*, if $D = 0$, solution should not be made for $\frac{A}{D}, \frac{B}{D}, \frac{C}{D}$, but for $\frac{A}{C}, \frac{B}{C}$ (say). (*b*) The equations may differ only by constant factors, then the three equations have an infinite number of solutions. This is explained by the fact that the points are on a straight line, and *any* plane through the line will pass also through the points.

216. The intercept equation of a plane. A plane will in general cut each coördinate axis at some definite distance from the origin, and this distance is called the **intercept** of the plane on the axis. If a, b, c be the intercepts on the x-, y-, and z-axes, respectively, of the plane whose equation is

$$Ax + By + Cz + D = 0, \quad . \quad . \quad . \quad . \quad (1)$$

then the points $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$ are points of the plane, and therefore (cf. Art. 215)

$$Aa + D = 0, \quad Bb + D = 0, \quad Cc + D = 0,$$

i.e.,

$$A = -\frac{D}{a}, \quad B = -\frac{D}{b}, \quad C = -\frac{D}{c}. \quad . \quad . \quad . \quad (2)$$

Hence equation (1) may be written

$$\frac{Dx}{a} + \frac{Dy}{b} + \frac{Dz}{c} - D = 0,$$

i.e.,

$$\boldsymbol{\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1}; \quad . \quad . \quad . \quad [17]$$

and this is the equation of the plane in terms of its intercepts.

217. The normal equation of a plane. A plane is wholly determined in position if the length and direction be known of a perpendicular to it from the origin; and this method of fixing a plane leads to one of the most useful forms of its equation. Let OQ be the perpendicular from the origin O to the plane ABC, let p be its length, always considered as positive, and let α, β, γ be its direction angles. Let $P \equiv (x, y, z)$ be any point of the plane, and draw its coördinates OM, MM', $M'P$. Then, projecting upon OQ,

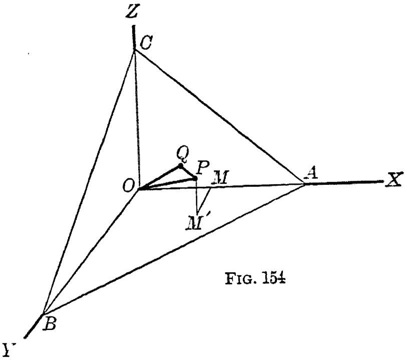


FIG. 154

$$\text{proj. } OMM'P = \text{proj. } OP,$$

hence $\text{proj. } OM + \text{proj. } MM' + \text{proj. } M'P = \text{proj. } OP,$

that is, $$x\cos\alpha + y\cos\beta + z\cos\gamma = p. \quad . \quad . \quad . \quad [18]$$

This is called the *normal* equation of the plane.

There are two special cases to be considered:

(1) If the plane is perpendicular to a coördinate plane, *e.g.*, to the xy-plane (cf. Art. 210), then $\gamma = 90°$, $\cos\gamma = 0$, and equation [18] reduces to

$$x\cos\alpha + y\cos\beta = p. \quad . \quad . \quad . \quad [19]$$

(2) If the given plane is parallel to one of the coördinate planes, *e.g.*, to the xy-plane (cf. Art. 209); then $\alpha = \beta = 90°$, $\gamma = 0°$, and eq. [17] reduces to

$$z = p. \quad . \quad . \quad . \quad [20]$$

218. Reduction of the general equation of first degree to a standard form.* **Determination of the constants $a, b, c, p, \alpha, \beta, \gamma$.** I. *Intercept form.* In Art. 216 a method has been indicated for reducing the general equation

$$Ax + By + Cz + D = 0 \quad . \quad . \quad . \quad (1)$$

to the intercept form. Since the points $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$ are on the plane (1), it follows that the intercepts are

$$a = -\frac{D}{A}, \quad b = -\frac{D}{B}, \quad c = -\frac{D}{C}. \quad . \quad . \quad . \quad (2)$$

II. *Normal form.* If equation (1) and the equation

$$x\cos\alpha + y\cos\beta + z\cos\gamma - p = 0 \quad . \quad . \quad . \quad (3)$$

represent the same plane, then their first members can differ only by a constant factor, m (cf. Art. 203, eqs. [5]; also Art. 58);

---

* The reduction of this article gives a second proof that the general algebraic equation of first degree always has for its locus a plane.

therefore

$$mA = \cos\alpha,\quad mB = \cos\beta,\quad mC = \cos\gamma,\quad mD = -p,$$

but, by [4], $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1,$

hence $m^2(A^2 + B^2 + C^2) = 1$, and $m = \dfrac{1}{\sqrt{A^2 + B^2 + C^2}}$.

Then
$$\left.\begin{aligned} \cos\alpha &= \frac{A}{\sqrt{A^2+B^2+C^2}}, & \cos\beta &= \frac{B}{\sqrt{A^2+B^2+C^2}},\\ \cos\gamma &= \frac{C}{\sqrt{A^2+B^2+C^2}}, & p &= \frac{-D}{\sqrt{A^2+B^2+C^2}}.\end{aligned}\right\} \quad [21]$$

Equation (1) written in the normal form is then

$$\frac{A}{\sqrt{A^2+B^2+C^2}}x + \frac{B}{\sqrt{A^2+B^2+C^2}}y + \frac{C}{\sqrt{A^2+B^2+C^2}}z = \frac{-D}{\sqrt{A^2+B^2+C^2}}; \quad . \quad . \quad . \quad (5)$$

therefore, to reduce equation (1) to the normal form, it is necessary only to *transpose the constant term to the second member of the equation, and then divide both members by the square root of the sum of the squares of the coefficients of the variable terms.* The sign of the radical is determined by the fact (Art. 217) that p is taken positive; hence, *the sign of the radical is the opposite of the sign of the constant term.*

219. The angle between two planes. Parallel and perpendicular planes. The angles formed by two intersecting planes are the same as the angles formed by two straight lines perpendicular to them respectively; *i.e.*, is the same as the angles between the respective normals from the origin to the planes. If

$$A_1x + B_1y + C_1z + D_1 = 0, \quad . \quad . \quad . \quad (1)$$

and

$$A_2x + B_2y + C_2z + D_2 = 0, \quad . \quad . \quad . \quad (2)$$

be two planes, then the direction cosines of their normals are respectively (eqs. [21])

$$\cos\alpha_1 = \frac{A_1}{\sqrt{A_1^2+B_1^2+C_1^2}}, \ \cos\beta_1 = \frac{B_1}{\sqrt{A_1^2+B_1^2+C_1^2}}, \ \cos\gamma_1 = \frac{C_1}{\sqrt{A_1^2+B_1^2+C_1^2}};$$

$$\cos\alpha_2 = \frac{A_2}{\sqrt{A_2^2+B_2^2+C_2^2}}, \text{ etc.,}$$

and by equation [4], if θ be the angle between the two planes, and hence between the two normals,

$$\cos\theta = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\ \sqrt{A_2^2 + B_2^2 + C_2^2}}. \quad . \quad . \quad . \quad [22]$$

There are two cases of special interest.

I. *Parallel planes.* If the planes (1) and (2) are parallel, their normals from the origin will have the same direction cosines, and differ only in length; therefore, by equations [20], the equations of the planes must be such that the coefficients of the variable terms are the same in the two equations, or can be made the same by multiplying one equation by a constant. In other words, if the planes (1) and (2) are parallel, then

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}; \quad . \quad . \quad . \quad [23]$$

and the plane

$$Ax + By + Cz + K = 0 \quad . \quad . \quad . \quad (3)$$

is parallel to the plane

$$Ax + By + Cz + D = 0, \quad . \quad . \quad (4)$$

for all values of the parameter K.

II. *Perpendicular planes.* If the planes (1) and (2) are perpendicular to each other, then $\cos\theta = 0$,

and $$A_1A_2 + B_1B_2 + C_1C_2 = 0; \quad . \quad . \quad . \quad [24]$$
and conversely.

220. Distance of a point from a plane. Let

$$P_1 \equiv (x_1, y_1, z_1)$$

be a given point, and

$$Ax + By + Cz + D = 0 \quad . \quad . \quad . \quad (1)$$

a given plane. The perpendicular distance of P_1 from the plane is equal to the distance from the plane (1) to a parallel plane through the point; *i.e.*, is equal to the difference in the lengths of the normals, from the origin, to these two parallel planes.

The parallel plane through P_1 has for its equation by Art. 219, equation (3),

$$Ax + By + Cz = Ax_1 + By_1 + Cz_1. \quad . \quad . \quad . \quad (2)$$

By [21], the lengths of the normals to planes (1) and (2) are, respectively,

$$p = \frac{-D}{\sqrt{A^2 + B^2 + C^2}}, \quad p' = \frac{Ax_1 + By_1 + Cz_1}{\sqrt{A^2 + B^2 + C^2}},$$

therefore if $d = p' - p$ be the required distance,

$$d = \frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}}. \quad . \quad . \quad . \quad [25]$$

In formula [25], the sign of the radical is taken opposite to the sign of D (Art. 218); and the sign of d shows on which side of the given plane lies the given point.

II. The Straight Line

221. Two equations of the first degree represent a straight line. Every equation of first degree represents a plane

(Art. 214), and two equations considered as simultaneous represent the intersections of their two loci (Art. 39). Therefore since two planes intersect in a straight line, the locus of the two simultaneous equations of first degree,

$$A_1x + B_1y + C_1z + D_1 = 0,\ A_2x + B_2y + C_2z + D_2 = 0, \ldots \quad (1)$$

is a straight line. As suggested in Art. 212, it is generally more simple to represent the straight line by equations in two variables only, *standard forms*, to which equation (1) can always be reduced.

222. Standard forms for the equations of a straight line. (*a*) *The straight line through a given point in a given direction.* Let $P_1 \equiv (x_1, y_1, z_1)$ be a given point, and α, β, γ the direction angles of a straight line through it. Let $P \equiv (x, y, z)$ be any point on the line, at a distance d from P_1. Then by equation [6],

$$d \cos\alpha = x - x_1,\ d\cos\beta = y - y_1,\ d\cos\gamma = z - z_1, \ldots \quad (1)$$

hence

$$\frac{x - x_1}{\cos\alpha} = \frac{y - y_1}{\cos\beta} = \frac{z - z_1}{\cos\gamma}; \quad . \quad . \quad . \quad [26]$$

which are the equations of a straight line in the first standard form, called the *symmetrical equations.*

(*b*) *The straight line through two given points.* Let $P_1 \equiv (x_1, y_1, z_1)$ and $P_2 \equiv (x_2, y_2, z_2)$ be the given points. Any straight line passing through P_1 has [26] for its equations. If the line passes also through P_2, then

$$\frac{x_2 - x_1}{\cos\alpha} = \frac{y_2 - y_1}{\cos\beta} = \frac{z_2 - z_1}{\cos\gamma}; \quad . \quad . \quad . \quad (2)$$

and hence from equations [26] and (2), by division to eliminate the unknown direction cosines,

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z1}{z_2 - z_1}. \quad . \quad . \quad . \quad [27]$$

These are the second standard forms for the equation of a straight line.

(*c*) *The straight line with given traces on the coördinate planes.* One of the simplest set of planes for determining a straight line is a pair of planes through the line and perpendicular respectively to the coördinate planes (cf. Art. 212). Then the equation of these planes will be the same as the equations of the traces of the line on the corresponding coördinate planes (Art. 210). Thus, if the equation of the traces of a given line upon the zx- and yz-planes are, respectively,

$$\left.\begin{aligned} x &= mz + b, \\ y &= nz + d, \end{aligned}\right\} \quad . \quad . \quad . \qquad [28]$$

then, considered as simultaneous, these are also the equations of the given line in space.

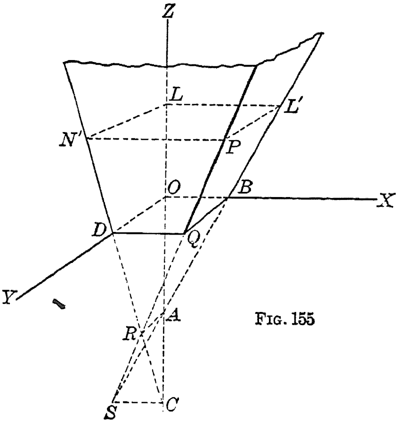


FIG. 155

In Fig. 155 the given traces are ABL' in the zx-plane, and CDN' in the yz-plane; P is any point in the given straight line, and Q, R, S are the points where the line pierces the xy-, yz-, zx-planes, respectively. Then it is clear that in equations [28]

$$\left.\begin{aligned} m &= \tan\angle\, OAB. & b &= OB, \\ n &= \tan\angle\, OCD, & d &= OD. \end{aligned}\right\} \quad . \quad . \quad . \qquad (3)$$

Also, since, by equations [28],

$$OA = -\frac{b}{m},\ AR = \frac{dm - bn}{m},\ OC = -\frac{d}{n},\ CS = \frac{bn - dm}{n},$$

therefore the points where the given line pierces the coördinate planes are

$$Q \equiv (b, d, 0),\ R \equiv \left(0, \frac{dm-bn}{m}, -\frac{b}{m}\right),\ S \equiv \left(\frac{bn-dm}{n}, 0, -\frac{d}{n}\right). \quad (4)$$

223. Reduction of the general equations of a straight line to a standard form. Determination of the direction angles and traces.

I. *Third standard form: traces.* The traces of a straight line have the same equations as have the planes of projection of the straight line upon the coördinate planes, respectively. They may be obtained, therefore (Art. 210), by eliminating in turn each of the variables z, y, x from the given equations.

This may be illustrated by a numerical example.

Given the equations

$$3x + 2y + z - 5 = 0,\ x + 2y - 2z = 3, \quad . \quad . \quad . \quad (1)$$

representing a straight line. Eliminating z, y, and x, successively, the equations

$$7x + 4y - 13 = 0,\ 2x + 3z - 2 = 0,\ 4y - 7z - 4 = 0 \ . \ . \ . \ (2)$$

are obtained, each representing a plane through the given line and perpendicular to a coördinate plane. Therefore these equations are also the equations of the traces of the line, in the xy-, zx-, and yz-planes, respectively.

II. *First standard form: direction angles.* The method of reducing the general equations of a straight line to the first standard form, and finding its direction angles, can also be illustrated by a numerical case.

Considering still the line whose equations are (1) above, and whose traces are given by equations (2); and taking the equations of any two of its traces, *e.g.*,

$$2x + 3z - 2 = 0, \quad 4y - 7z - 4 = 0; \quad . \quad . \quad . \quad (3)$$

these have one variable, z, in common. Equating the values of this common variable from the two equations, gives

$$z = \frac{-2x+2}{3} = \frac{4y-4}{7},$$

which may be written, to correspond with equations [26],

$$\frac{z-0}{1} = \frac{x-1}{-\frac{3}{2}} = \frac{y-1}{\frac{7}{4}}. \quad . \quad . \quad . \qquad (4)$$

Now, although the denominators $1, -\frac{3}{2}, \frac{7}{4}$ of equation [4] are not direction cosines of any line, yet, by equations [5], they differ from such direction cosines only by the factor

$$\sqrt{1 + \tfrac{9}{4} + \tfrac{49}{16}} = \tfrac{1}{4}\sqrt{101}.$$

Rewriting equations (4) in the form

$$\frac{x-1}{\frac{-6}{\sqrt{101}}} = \frac{y-1}{\frac{7}{\sqrt{101}}} = \frac{z-0}{\frac{4}{\sqrt{101}}}, \quad . \quad . \quad . \qquad (5)$$

it corresponds entirely to equations [26]. Therefore the line passes through the point (1, 1, 0), and its direction angles are given by the relations

$$\cos\alpha = -\frac{6}{\sqrt{101}},\quad \cos\beta = \frac{7}{\sqrt{101}},\quad \cos\gamma = \frac{4}{\sqrt{101}}.$$

The method given above is evidently perfectly general.

224. The angle between two lines; between a plane and a line. If the equations of two straight lines be written in the form

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}, \quad . \quad . \quad . \qquad (1)$$

$$\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}, \quad . \quad . \quad . \qquad (2)$$

then by Art. 223, II, their direction cosines are, respectively,

$$\cos\alpha_1 = \frac{a_1}{\sqrt{a_1^2+b_1^2+c_1^2}}, \qquad \cos\alpha_2 = \frac{a_2}{\sqrt{a_2^2+b_2^2+c_2^2}},$$

$$\cos\beta_1 = \frac{b_1}{\sqrt{a_1^2+b_1^2+c_1^2}}, \text{ etc.}, \quad . \quad . \quad . \qquad (2)$$

and therefore, by equation [10], the angle between the two lines is given by the equation

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}. \quad . \; . \; . \quad [29]$$

Again, the angle between the straight line

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}, \quad . \; . \; . \quad (3)$$

and the plane

$$Ax + By + Cz + D = 0 \quad . \; . \; . \quad (4)$$

is the complement of the angle between the line (3) and the perpendicular to the plane (4) from the origin. Therefore, by equations [10] and [21], and Art. 223, II, the required angle is given by the equation

$$\sin\theta = \frac{aA + bB + cC}{\sqrt{a^2 + b^2 + c^2}\sqrt{A^2 + B^2 + C^2}}. \quad . \; . \; . \quad [30]$$

Conditions for perpendicularity and parallelism precisely like those of Art. 219 may be obtained from equations [29] and [30].

EXAMPLES ON CHAPTER III

1. Find the equation of a line through the points (1, 2, 3) and (3, 2, 1).

2. Find the equation of a plane through three points (1, 2, 3), (3, 2, 1), and (2, 3, 1).

3. Write the equations of the straight line through the point (1, 2, 3), and having its direction cosines proportional to $\sqrt{3}$, 1, $2\sqrt{3}$.

4. What are the traces of the line of Ex. 1 upon the coördinate planes? Where does the line pierce those planes?

5. Find the equations of a straight line through the point (1, 2, 3) and perpendicular to the plane $x + 2y + 3z = 6$.

Reduce to the intercept and normal forms, and determine which octant each plane cuts:

6. $2x - 3y - z = 7$; **7.** $5y + 2z - 1 = x$.

8. Reduce the equations of the line

$$2x - 3y - z = 7, \quad 5y + 2z - 1 = x$$

to the symmetrical form, and determine its direction cosines.

9. Find the angle between the planes

$$2x - 3y - z = 7, \quad 5y + 2z - 1 = x.$$

10. Find the angle between the line

$$x + y + 2z = 0, \quad 2x - y - 2z - 1 = 0,$$

and the plane $3x + 6z - 5y + 1 = 0.$

11. Write the equation of a plane parallel to the plane

$$2x - y + 7z - 5 = 0,$$

and passing through the point (0, 0, 0); through the point ($^-1$, 1, $^-1$).

12. Write the equation of a plane perpendicular to the plane

$$3x + 5y - z + 6 = 0,$$

and passing through the two points (3, 1, 2) and (0, $^-2$, $^-4$).

13. Find the distances of the points (7, $^-2$, 3) and (3, 3, 1) from the plane $2x + 5y - z - 9 = 0$. Are they on the same side of the plane?

14. At what angle does the plane $ax + by + cz + d = 0$ cut each coördinate plane? Each coördinate axis?

15. Find the equation of a plane through the point (1, 1, 1) and perpendicular to each of the planes

$$2x - 3y + 7z = 1, \quad x - y - 2z = 2.$$

16. Write the equation of a plane whose distance from the point (0, 2, 1) is 3, and which is perpendicular to the radius vector of the point (2, $^-1$, $^-1$).

17. Write the equation of a straight line through the point (5, 2, 6) which is parallel to the line

$$2x - 3z + y - 2 = 0, \quad x + y + z + 1 = 0.$$

18. Find the traces on the coördinate planes of the line

$$2x - 3z + y - 2 = 0, \quad x + y + z + 1 = 0.$$

19. Prove that the planes

$$2x - 3y + z + 1 = 0,$$
$$5x + z - 1 = 0,$$
$$19x - 3y - 4z - 5 = 0,$$

have one line in common.

20. What is the equation of the plane determined by the line

$$2x - 3z + y - 2 = 0, \quad x + y + z + 1 = 0,$$

and the point (5, 2, 6)?

21. Show analytically that the locus of a point equidistant from three given points is a straight line perpendicular to the plane determined by those three points.

22. Derive equation [17] directly from a figure, without using equation [16].

CHAPTER IV

EQUATIONS OF THE SECOND DEGREE

QUADRIC SURFACES

225. The locus of an equation of second degree. The most general algebraic equation of second degree in three variables may be written

$$\begin{aligned}&Ax^2 + By^2 + Cz^2 + 2\,Fyz + 2\,Gxz + 2\,Hxy + 2\,Lx + 2\,My \\ &\quad + 2\,Nz + K = 0. \quad . \quad . \quad . \qquad [31]\end{aligned}$$

Any surface which is the locus of an equation of second degree is called a **quadric** surface, and is of particular interest because of its close connection with and analogy to the conic sections. In fact, every plane section of a quadric is a conic, as may be easily shown as follows.

By Art. 207, any plane may be chosen as a coördinate plane, and the transformation of coördinates to the new axes will leave the degree of equation [31] unchanged; *i.e.*, the new equation of the locus will still be of the form [31], though with different values for the coefficients. To find the nature of any plane section, choose the given plane as (say) the xy-plane of reference, and transform to the new axes; the new equation will be of form (1). Then let $z = 0$. The equation of the section of the quadric is

$$Ax^2 + By^2 + 2\,Hxy + 2\,Lx + 2\,My + K = 0; \; . \; . \quad (1)$$

and this, by Art. 175, represents a conic.

Moreover, the trace of the surface on any parallel plane, as $z = a$, is given by the equation

$$\begin{aligned} Ax^2 + By^2 + 2\,Hxy + 2(L + ab)x + 2(M + aF)y \\ + (Ca^2 + 2\,Ma + K) = 0. \quad . \quad . \quad . \quad (2) \end{aligned}$$

Now, by Arts. 177, 181, the loci of equations (1) and (2) are conics of the same species, and with semi-axes proportional; therefore their eccentricities are equal, and the curves are similar. Hence, *all parallel plane sections of a quadric are similar conics.*

226. Species of quadrics. Simplified equation of second degree. As will be seen in the following sections, quadric surfaces may be conveniently classed under four species. For, although different plane sections of any surface will in general be conics of different species, still the *general* form of the surface may be characterized most strikingly by those plane sections which are ellipses, hyperbolas, parabolas, or straight lines. These species are called, respectively, *ellipsoids*, *hyperboloids*, *paraboloids*, and *cones;* and each species has special varieties, depending upon the nature of a second system of plane sections. To study these species it will be well to simplify the general equation of second degree as much as possible by a suitable transformation of coördinates.*

A transformation of coördinates changing to a new rectangular system having the same origin as the old, by equations [14], will transform the given equation of second degree to

$$\begin{aligned} A'x^2 + B'y^2 + C'z^2 + 2\,F'yz + 2\,G'xz + 2\,H'xy + 2\,L'x \\ + 2\,M'y + 2\,N'z + K = 0, \quad . \quad . \quad . \quad (1) \end{aligned}$$

where A', B', $\cdots$ N' are functions of the nine direction angles

---

* Compare with Art. 175.

$\alpha_1, \alpha_2, \cdots$ of the new axes, which are limited by the six independent equations noted in Art. 207. These angles, therefore, may be so chosen that three additional conditions shall be fulfilled; hence, so that the coefficients F', G', and H' shall vanish. Then the new equation of the quadric will be

$$A'x^2 + B'y^2 + C'z^2 + 2L'x + 2M'y + 2N'z + K = 0. \quad (2)$$

Now a second transformation may be made to a parallel system of axes through a new origin (h, k, j), by equations [13], giving for the new equation

$$A'x^2 + B'y^2 + C'z^2 + 2L''x + 2M''y + 2N''z + K' = 0, \quad (3)$$

in which L'', M'', N'', and K' are functions of the coördinates h, k, and j; and these coördinates may be chosen so that L'', M'', and N'' will vanish, giving for the simplified form of the equation of the given quadric,

$$A'x^2 + B'y^2 + C'z^2 + K' = 0. \quad . \quad . \quad . \quad (4)$$

It may happen, however, that the choice given above for the direction angles $\alpha_1, \alpha_2, \cdots$, of the new axes is such that the coefficient of one more term of second degree, as C', will also vanish; then equation (4) would reduce to

$$A'x^2 + B'y^2 + K' = 0, \quad . \quad . \quad . \quad (5)$$

and the surface is a cylinder (Art. 210). Again, if also L'', M'', N'' are not independent, and the values of h, k, j as given above are therefore indeterminate, then h, k, j may be chosen so that, for example, L'', M'', and K' shall vanish; and the equation of the quadric becomes

$$A'x^2 + B'y^2 + 2N''Z = 0.^* \quad . \quad . \quad . \quad (6)$$

---

* If the coefficients of two quadratic terms vanish, as B' and C', a change of origin first, then of direction of axes, may be chosen so that the equation will reduce to the form (6).

The two forms of the quadric, not already discussed,* have therefore for their equations, when simplified (dropping the accents),

$$Ax^2 + By^2 + Cz^2 + K = 0, \quad . \quad . \quad . \quad [32]$$

and

$$Ax^2 + By^2 + 2Nz = 0. \quad . \quad . \quad . \quad [33]$$

A center of a surface is a point such that it bisects every chord of the surface which passes through it. It is clear that the locus of equation [32] is a *central* quadric, while the locus of equation [33] is *non-central* (cf. Art. 178).

227. Standard forms of the equation of a quadric. For convenience of discussion, the intercepts of the locus of equation [32] on the coördinate axes may be represented by a, b, c, respectively, so that

$$a^2 = -\frac{K}{A}, \quad b^2 = -\frac{K}{B}, \quad c^2 = -\frac{K}{C}. \quad . \quad . \quad . \quad (1)$$

Then, since A, B, C, and K cannot be all of the same sign, there will be three types of equation [32], according to the signs of A, B, C, and K; viz.:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad . \quad . \quad . \quad (2)$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \quad . \quad . \quad . \quad (3)$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1. \quad . \quad . \quad . \quad (4)$$

Similarly, equation [33] may be written for convenience in the typical forms

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = z, \quad . \quad . \quad . \quad (5)$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = z, \quad . \quad . \quad . \quad (6)$$

---

* An exceptional case occurs where the general equation can be factored into linear factors, and therefore represents two planes.

wherein, however, a and b are no longer intercepts as in (2), (3), and (4).

Again, if the equation [32] has its constant term zero, it may be written in two typical forms,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0, \quad . \quad . \quad . \quad (7)$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0. \quad . \quad . \quad . \quad (8)$$

These seven equations are *standard forms* of the equation of second degree, and will be discussed in turn.

228. The ellipsoid: equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. From the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad . \quad . \quad . \quad [34]$$

the following properties of its locus may be derived:

(1) The traces on each coördinate plane are ellipses, having

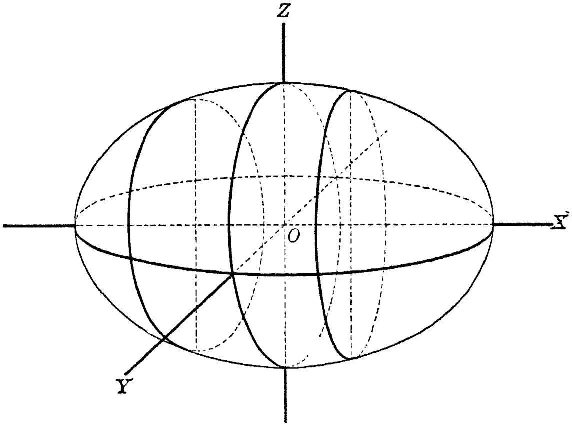


FIG. 156

the semi-axes a and b in the xy-plane, b and c in the yz-plane, and c and a in the zx-plane.

(2) The traces on planes parallel to any coördinate plane are similar ellipses (Art. 225).

(3) The equation may be written

$$\frac{y^2}{\frac{b^2(a^2-x^2)}{a^2}}+\frac{z^2}{\frac{c^2(a^2-x^2)}{a^2}}=1;$$

hence for a plane section parallel to the yz-plane, the semi-axes are real if the value of x lies between $-a$ and $+a$, imaginary if beyond those limits, and zero if $x=\pm a$. Moreover, the length of the axes diminish continuously from the values b and c, respectively, when $x=0$ to the value zero, when $x=\pm a$.

Similarly for sections parallel to either of the other coördinate planes.

(4) The surface is symmetrical with respect to each coördinate plane.

This quadric surface, the locus of equation [34], is called an **ellipsoid**. It may be conceived as generated by a variable ellipse, which has its vertices upon, and moves always perpendicular to, two fixed ellipses, which in turn are perpendicular to each other and have one axis in common.

From this definition equation [34] can be easily derived. Let CRA and ASB be fixed ellipses perpendicular to each other, and having the semi-axis OA in common, and the second axes OC and OB, respectively; and let SPR be the variable ellipse, with semi-axes MS and MR. If OA, OB, OC be taken as the x, y, z axes, respectively; and P be any point on the moving ellipse, with coördinates OM, MM', $M'P$, then (by Art. 112),

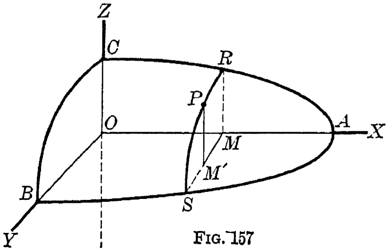


FIG. 157

$$\frac{\overline{M'P}^2}{\overline{MR}^2}+\frac{\overline{MM'}^2}{\overline{MS}^2}=1, \qquad \frac{\overline{MR}^2}{\overline{OC}^2}+\frac{\overline{OM}^2}{\overline{OA}^2}=1, \qquad \frac{\overline{MS}^2}{\overline{OB}^2}+\frac{\overline{OM}^2}{\overline{OA}^2}=1,$$

i.e., $\frac{z^2}{\overline{MR}^2}+\frac{y^2}{\overline{MS}^2}=1$, (1) $\quad \frac{\overline{MR}^2}{c^2}+\frac{x^2}{a^2}=1$, (2) $\quad \frac{\overline{NS}^2}{b^2}+\frac{x^2}{a^2}=1$. (3)

By equations (2) and (3).

$$\overline{MR}^2=c^2\left(1-\frac{x^2}{a^2}\right), \qquad \overline{MS}^2=b^2\left(1-\frac{x^2}{a^2}\right).$$

Substitution in (1) gives $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

Every algebraic equation of the form

$$Ax^2+By^2+Cz^2-K=0$$

represents an ellipsoid. If two of the coefficients of the variable terms are equal it is an ellipsoid of revolution, either an oblate or prolate spheroid; and if the three coefficients of the variable terms are equal, it is a sphere (cf. Art. 213, eqs. (10), (11), and (8)).

229. The un-parted hyperboloid: equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$. From the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \qquad . \quad . \quad . \qquad [35]$$

the following properties of its locus may be derived:

(1) The trace on the xy-plane is an ellipse, with semi-axes a and b; while the traces on the yz- and zx planes are hyperbolas, having the semi-axes b and c, c and a, respectively, and the conjugate axes along the z-axis.

(2) The traces on planes parallel to any coördinate plane are similar conics, ellipses or hyperbolas, respectively (Art. 225).

(3) The traces on the planes $x = a$, $x = -a$, $y = b$, $y = -b$ are in each case a pair of intersecting straight lines.

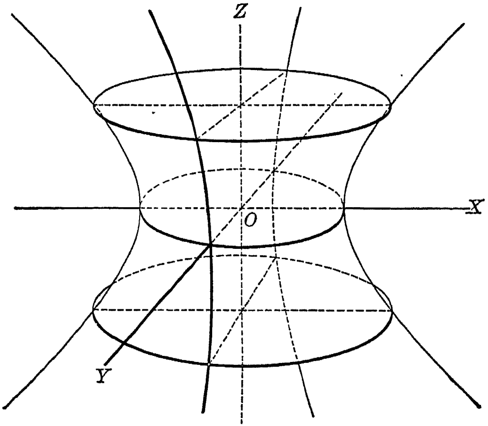


FIG. 158

(4) The equation may be written

$$\frac{x^2}{\dfrac{a^2(c^2+z^2)}{c^2}} + \frac{y^2}{\dfrac{b^2(c^2+z^2)}{c^2}} = 1, \quad . \quad . \quad . \quad (1)$$

or

$$\frac{y^2}{\dfrac{b^2(a^2-x^2)}{a^2}} - \frac{z^2}{\dfrac{c^2(a^2-x^2)}{a^2}} = 1. \quad . \quad . \quad . \quad (2)$$

From equation (1) it appears that the trace on the xy-plane is the smallest of the system of ellipses parallel to that plane, and that the sections increase continuously and indefinitely as z increases from 0 to $\pm\infty$.

From equation (2) it appears that the transverse axis of the hyperbolas parallel to the yz-plane is parallel to the y-axis. Similarly for the xz-sections the transverse axis is parallel to the x-axis.

(5) The surface is symmetrical with respect to each coordinate plane.

This quadric surface, whose equation is [35], is called an **un-parted hyperboloid,** or an **hyperboloid of one sheet.** It may be conceived as generated by a variable ellipse, which has its vertices upon and moves always perpendicular to two fixed hyperbolas, which in turn are perpendicular to each other, and have a common conjugate axis. Its equation can be readily obtained from this definition.*

Every equation of the form $Ax^2 + By^2 - Cz^2 - K = 0$ represents an un-parted hyperboloid. If the two positive coefficients are equal, *i.e.*, if $a = b$, the quadric is the simple hyperboloid of revolution (Art. 213, eq. (12)).

230. The bi-parted hyperboloid: equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$ From the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad . \quad . \quad . \qquad [36]$$

the following properties of its locus may be derived:

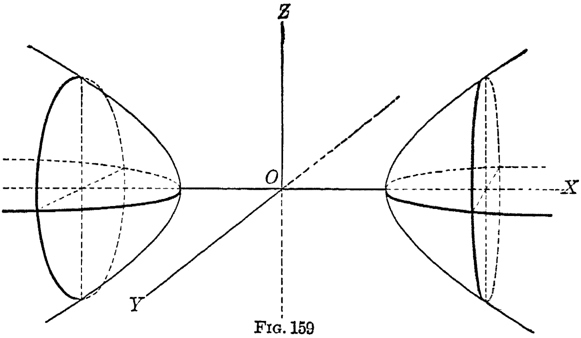


FIG. 159

---

*Cf. Art. 227.

(1) The traces on the xy- and zx-planes are hyperbolas, with semi-axes a and b, c and a, respectively, and with the transverse axis along the x-axis, while the traces on the planes parallel to the yz-plane are imaginary if x lies between a and $-a$, real ellipses if x is beyond those limits, and points if $x = \pm a$.

(2) The traces on planes parallel to any coördinate plane are similar (Art. 225).

(3) The elliptical sections parallel to the yz-plane increase continuously and indefinitely as x varies from $+a$ to $+\infty$, or from $-a$ to $-\infty$.

(4) The surface is symmetrical with respect to each coördinate plane.

This quadric surface, whose equation is [36], is called a **bi-parted hyperboloid,** or **hyperboloid of two sheets.** It may be conceived as generated by a variable ellipse which has its vertices upon, and moves always perpendicular to, two fixed hyperbolas which in turn are perpendicular to each other, and have a common transverse axis. This definition leads readily to the equation [36].

Every equation of the form $Ax^2 - By^2 - Cz^2 - K = 0$ represents a bi-parted hyperboloid. If the coefficients of the two negative variable terms are equal, *i.e.*, if $b = c$, the surface is the double hyperboloid of revolution (cf. Art. 213, eq. (13)).

231. **The paraboloids: equation $\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = z$.** A discussion of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = z \quad \cdot \quad \cdot \quad \cdot \quad [37]$$

similar to that of the preceding articles shows that its locus

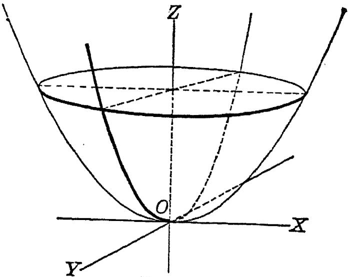


FIG. 160

is as represented in Fig. 160, symmetrical with respect to the *yz*- and *zx*-plane, but not with respect to the *xy*-plane. This quadric is the **elliptic paraboloid**, and may be conceived as being generated by a variable parabola which has its vertex upon, and moves always perpendicular to, a fixed parabola, the axes of the two parabolas being parallel and lying in the *same* direction. This definition leads directly to equation [37].*

Every equation of the form $Ax^2 + By^2 - 2Nz = 0$ represents an elliptic paraboloid. If the two positive coefficients are equal, the quadric is a paraboloid of revolution (cf. Art. 213, eq. (14)).

Similarly, the equation $\frac{x_2}{a^2} - \frac{y_2}{b_2} = z$ . . . . [38]

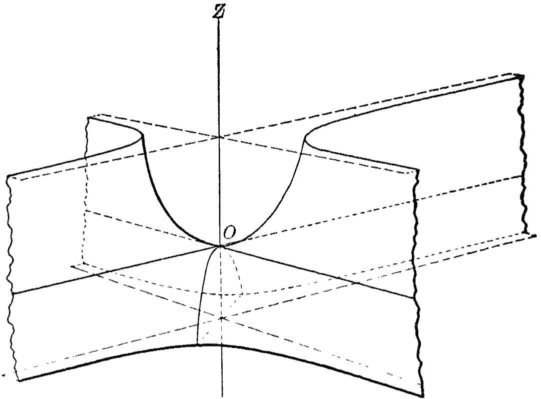


FIG. 161

---

* See Art. 228.

has for its locus a surface as represented in Fig. 161. This quadric is the **hyperbolic paraboloid**, and may be conceived as generated by a variable parabola which has its vertex upon and moves always perpendicular to a fixed parabola, the axes of the two parabolas being parallel, but lying in *opposite* directions. Equation [38] may be derived at once from this definition.*

Every equation of the form $Ax^2 - By^2 - 2Nz = 0$ represents an hyperbolic paraboloid.

232. The cone: equation $\frac{x_2}{a_2}+\frac{y_2}{b_2}-\frac{z_2}{c_2}=0.$

The equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0$ evidently is satisfied by the coördinates of only one real point, viz. the origin. No further discussion of this equation is necessary. But the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0 \quad . \quad . \quad [39]$$

has a locus of importance, having the following properties:

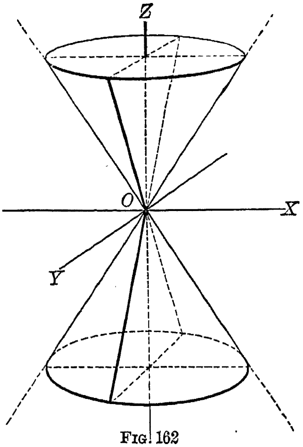


Fig. 162

(1) The origin is a point of the locus.

(2) The trace on the xy-plane is a point. The traces on planes parallel to the xy-plane are similar ellipses, whose semi-axes increase continuously and indefinitely as z increases from 0 to ± ∞.

(3) The trace on each of the other coördinate planes is a pair of straight lines which intersect at the origin.

---

* See Art. 228.

(4) The surface is symmetrical with respect to each coördinate plane, hence also with respect to the origin.

(5) The straight line through the origin and any other point of the locus lies wholly in the locus.

This quadric surface is called a **cone,** and the origin is its **vertex.** It may be conceived as generated by a straight line which moves along a fixed ellipse as directrix, and passes through a fixed point in a straight line which is perpendicular to the plane of the ellipse at its center.

Every equation of the form $Ax^2 + By^2 - Cz^2 = 0$ represents a cone. If the two positive coefficients are equal, it is a cone of revolution, or circular cone (cf. Art. 213, eq. (9)).

The reasoning of Art. 225, applied to the special equation of the form [31] which represents a cone, gives an analytic proof of the fact that every plane section of a cone is a second degree curve (cf. Art. 48; Appendix, Note D).

233. The hyperboloid and its asymptotic cone. The hyperboloid

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

and the cone

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

are closely related. It is clear that, since the equations differ only in the constant terms, the surfaces can have no finite points in common; while as the values of y and z are increased indefinitely, the corresponding values for x from the two equations be-

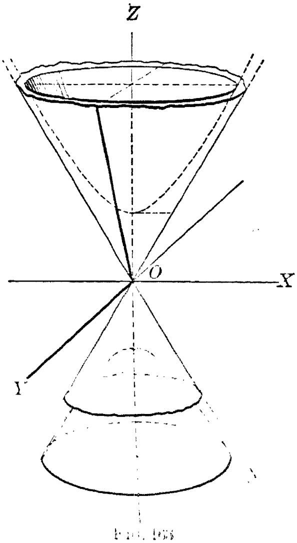


FIG. 153

come relatively nearer. In fact, the hyperboloid may be said to be tangent to the cone at infinity, and bears to the cone a relation entirely analogous to that between the hyperbola and its asymptotes. In the same way, the cone $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$ is asymptotic to the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$.

EXAMPLES ON CHAPTER IV

1. Derive the equation [35] directly from the definition of Art. 229.

2. Derive the equation [36] directly from the definition of Art. 230.

3. Derive the equations [37], [38] directly from the definitions of Art. 231.

4. Derive the equation [39] directly from the definition of Art. 232

5. Show analytically that the intersection of two spheres is a circle.

6. Find the equation of the tangent plane to the sphere $(x-a)^2+(y-b)^2+(z-c)^2=r^2$, at any point of the sphere.

7. Show that the equation $Ax_1x+By_1y+Cz_1z+K=0$ represents a plane tangent to the conic, $Ax^2+By^2+Cz^2+K=0$, at the point (x_1, y_1, z_1) on the quadric.

8. Find the equation of the cone with origin as vertex and the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ in the plane $z=-2$, as directrix.

9. Find the equation of a sphere having the line from $P_1 \equiv (x_1, y_1, z_1)$ to $P_2 \equiv (x_2, y_2, z_2)$ as a diameter.

10. Show that a sphere is determined by four points in space.

Write the equation of the quadric whose directing curves have the equations:

11. $\frac{x^2}{2}+\frac{y^2}{3}=1$, and $\frac{y^2}{3}+\frac{z^2}{9}=1$.

12. $\frac{x^2}{9}-\frac{y^2}{4}=1$, and $\frac{x^2}{9}-\frac{z^2}{16}=1$.

13. $\frac{x^2}{9}-\frac{y^2}{4}=1$, and $\frac{y^2}{4}-\frac{z^2}{16}=1$.

14. $z^2=16x$, and $y^2=9x$.

15. $x^2-4y=0$, and $z^2+3y=0$.

APPENDIX

NOTE A

Historical sketch.* Analytic Geometry, in the form in which it is now known, was invented by René Descartes (1596–1650) and first published by him in 1637, in the third section of a treatise on universal science entitled "Discours de la méthod pour bien conduire sa raison et chercher la verité dans la sciences." He made the invention while attempting to solve a certain problem, proposed by Pappus, the most important case of which is: to find the locus of a point such that the product of the perpendiculars drawn from it upon m given straight lines shall bear a constant ratio to the product of the perpendiculars drawn from it upon n other given straight lines. By pure geometry this problem had already been solved for the special cases when $m = 1$ and $n = 1$ or 2. Pappus had also asserted, but without proof, that when $m = n = 2$, then the locus of this point is a conic. In his effort to prove this fact Descartes introduced his system of coördinates and found the equation of the locus to be of the second degree, thus proving that it is a conic.

Analytic geometry does not consist merely (as is sometimes loosely said) in the application of algebra to geometry: that had been done by Archimedes and many others, and had become the usual method of procedure in the works of mathematicians of the sixteenth century. But in all these earlier applications a special set of axes were required for each individual curve. The great advance made by Descartes was that he saw that a point could be completely determined if its distances, say x and y, from two fixed lines, drawn at right angles to each other, in the plane, were given: and that though an equation $f(x, y)=0$ is indeterminate and can be satisfied by an infinite number of values of x and y, yet these values of x and y determine the coördinates of a number of points which form a curve of which the equation $f(x, y) = 0$ expresses some geometric property, *i.e.*, a property true for every point of the curve. Moreover, he saw that this method enables one to refer all the curves that may be under investigation to the *same* set of axes; and that in order to investigate the properties of a curve it is sufficient to select any characteristic geometric property, as a definition, and to express it as an equation by means of the (current) coördinates of any point on the curve; *i.e.*, to translate the definition into the language of analytic geometry — the equation so obtained contains implicitly every property of the curve, and any particular property can be deduced from it by ordinary algebra.

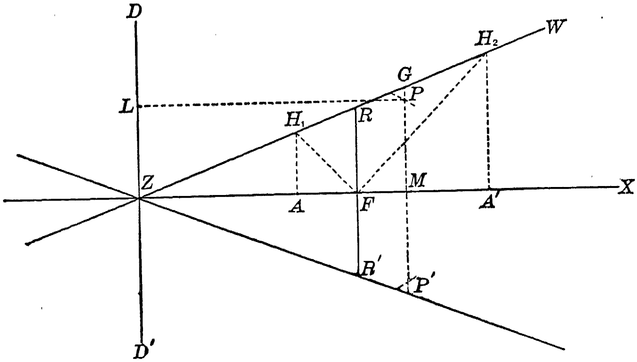
---

* Taken chiefly from Ball's History of Mathematics.

While the earlier geometry is an admirable instrument for intellectual training, and while it frequently affords an elegant demonstration of some proposition the truth of which is already known, it requires a special procedure for each individual problem; on the other hand, analytic geometry lays down a few simple rules by which any property can be at once proved. It is incomparably more potent than the geometry of the ancients for all purposes of research.

NOTE B

Construction of any conic, given directrix, focus, and eccentricity. Let $D'D$ be the directrix, F the focus, and e the eccentricity of a conic (cf. Part I, Art. 48), to plot the curve.



Construction: Draw ZFX perpendicular to $D'D$, and ZW so that, if $\alpha = \angle XZW$, $\tan\alpha = e$. Now draw FR perpendicular to ZF, cutting ZW at R; then R is a point of the conic; it is the end of the latus rectum.

Bisect the right angles at F by FH_1 and FH_2, intersecting ZW in H_1 and H_2, and draw H_1A and H_2A' perpendicular to ZX; then A and A' are points on the curve; they are the vertices of the conic.

Again, from any point G between H_1 and H_2 on ZW, draw MG perpendicular to ZX, cutting it at M; and from F as a center with MG as radius describe an arc cutting MG at P. Then P is a point of the curve.

Proof: for the point R, $\dfrac{FR}{ZF} = \tan\alpha = e$;

for the point A, $\dfrac{AF}{ZA} = \dfrac{AH_1}{ZA} = \tan\alpha = e$; $[\angle AFH_1 = 45^\circ]$

for the point P, $\dfrac{FP}{ZM} = \dfrac{MG}{ZM} = \tan\alpha = e$;

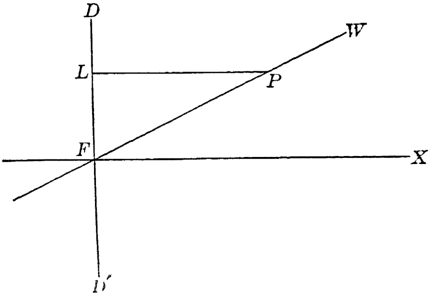
hence the points R, A_1, and P are such that their distances from the directrix and from the focus are in the ratio e; and each is therefore, according to the definition given in Art. 48, a point of the conic. By plotting various points P (and the symmetrical points P') and connecting them by a smooth curve, the conic may be plotted to any required degree of accuracy.

If $\alpha < 45^\circ$, then $\tan\alpha < 1$, *i.e.*, $e < 1$, and the conic is an ellipse; if $\alpha = 45^\circ$, the conic is a parabola; and if $\alpha > 45^\circ$, the conic is an hyperbola (cf. Part I, Art. 48).

NOTE C

The special cases of the conics. The locus of the second degree curve has been seen to have three species, according as $e < 1$, $e = 1$, or $e > 1$.

If $e = 0$, then, since b is defined by the equation $b^2 = a^2(1 - e^2)$, $b = a$. and the curve is an ellipse with equal axes, *i.e.*, it is a circle; in this case, also, the directrix is at infinity and the focus at the center, for the equation of the directrix is $x = \dfrac{a}{e}$, and the distance from the center to the focus is ae (cf. Part I, Arts. 110, 116).



Again, suppose the focus F to be on the directrix. Then, if P is any point of the locus, and LP perpendicular to FD,

$$FP = e \cdot LP, \quad . \quad . \quad . \qquad (1)$$

and

$$\sin \angle PFL = \frac{LP}{FP} = \frac{1}{e}; \quad . \quad . \quad . \qquad (2)$$

hence the angle PFL is constant, with two supplementary values for a given value of e.

The locus consists therefore of two straight lines intersecting at F, and equation (2) shows that:

if $e > 1$, the lines are real and different;

if $e = 1$, the lines are real and coincident;

and if $e < 1$, the lines are imaginary, and the real part of the locus consists of the point F.

Suppose now the directrix, with the focus upon it, to be at infinity; then, if $e > 1$, the locus is a pair of parallel lines.

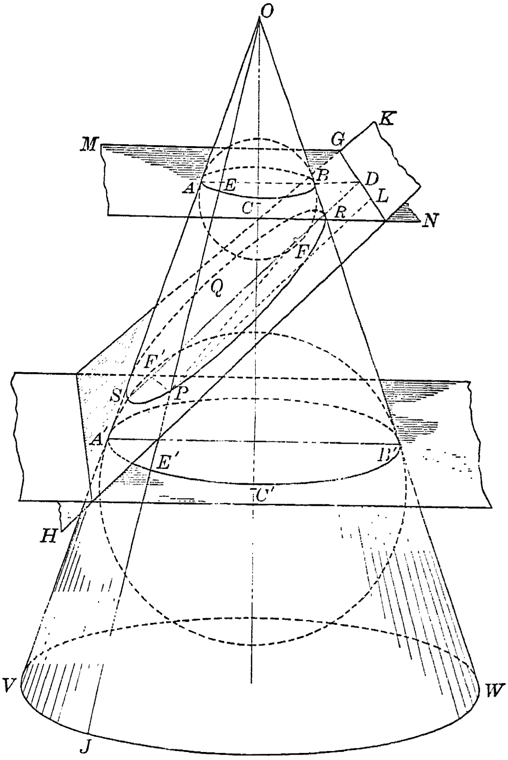
These results agree with those already summarized in Art. 182.

NOTE D

Sections of a cone made by a plane. The following proposition is due to Hamilton, Quételet, and others (see Taylor's Ancient and Modern Geometry of Conics, p. 204).

If a right circular cone is cut by a plane, and two spheres are inscribed in the cone and tangent to this plane, then the section of the cone made by the plane is a second degree curve (cf. Part I, Arts. 48, 175), of which the foci are the points of contact of the spheres and the plane, and the directrices are the lines in which this plane intersects the planes of the circles of contact of the spheres and the cone.

Construction: Let $O\text{–}VW$ be a right circular cone cut by the plane HK in the section $RPSQ$, P being any point of the section. Inscribe two spheres, $C\text{–}ABF$ and $C'\text{–}A'B'F'$, whose circles of contact with the cone are AEB and $A'E'B'$, respectively, and which are tangent to the plane HK in the points F and F'. Through P draw the element OP of the cone, cutting the circles of contact in the points E and E'. Also pass a plane MN through the circle AEB, and therefore perpendicular to the axis OCC' of the cone; it will intersect the plane HK in a straight line GDL, which is perpendicular to the straight line $F'F$. Draw PL perpendicular to GDL.



Then PL makes a constant angle θ ($\equiv \angle F'DA$) with the plane MN [since PL is parallel to $F'F$], and, if p represents the distance from the point P to the plane MN,

$$p = PL \sin \theta. \qquad . \quad . \quad . \qquad (1)$$

Also PE, being an element of the cone, makes a constant angle α with the plane MN

$$p = PE \sin \alpha. \qquad . \quad . \quad . \qquad (2)$$

Again, since tangents from an external point to a sphere are equal,

$$PE = PF. \qquad . \quad . \quad . \qquad (3)$$

Hence, from equations (1), (2), and (3)

$$\frac{PF}{PL} = \frac{\sin\theta}{\sin\alpha} = e, \text{ a constant,} \quad . \quad . \quad . \quad (4)$$

i.e., the ratio $PF:PL$, for every point P of the section $SPRQ$, is constant, and (Part I, Arts. 48, 175) the section is a second degree curve, with a focus at F, directrix GDL, and eccentricity $\frac{\sin\theta}{\sin\alpha}$.

Similarly, F' is the other focus, and the line of intersection of the planes HK and $A'E'B'$ is the other directrix of the conic $SPRQ$; hence the theorem is established.

Moreover, the plane VW, being perpendicular to the axis of the cone, and OVW, being a section made by a plane passing through the axis, $\alpha = \angle OVW$, and is constant for a given cone, while $\theta = \angle OSR$, and varies only with the plane HK.

Hence the eccentricity varies with the inclination of the plane HK, and there are the three following cases:

if $\theta < \alpha$, then $e < 1$, and the section is an ellipse;

if $\theta = \alpha$, then $e = 1$, and the section is a parabola;

if $\theta > \alpha$, then $e > 1$, and the section is an hyperbola.

Again, if the cutting plane HK passes through the vertex O of the cone, then the focus F is on the directrix GDL, and the section will be either a pair of straight lines or a point:

if $\theta < \alpha$, the section is a point, the vertex O of the cone.

if $\theta = \alpha$, the section is a pair of coincident straight lines, an element of the cone;

if $\theta > \alpha$, the section is a pair of intersecting straight lines, two elements through the vertex (cf. Note C).

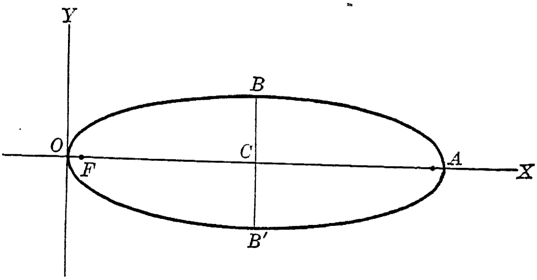
It is, of course, evident that for every elliptic section of the focal spheres both lie in the same nappe of the cone, and touch the plane of the section (HK) on opposite sides; while for every hyperbolic section these focal spheres lie one in each nappe of the cone, and both on the same side of the plane of the section.

In the above proof, for the sake of simplicity, a right circular cone was employed; it is easy to show (see Salmon's Conic Sections, p. 329) that every section of a second degree cone (right or oblique) by a plane is a second degree curve.

The demonstration just given shows also that the parabola is a limiting case of an ellipse (cf. Note E).

NOTE E

Parabola the limit of an ellipse,* or of an hyperbola. If a vertex and the corresponding focus of an ellipse remain fixed in position while the center moves further and further away, the major axis becoming infinitely long, then the form of the ellipse approaches more and more nearly to that of a parabola having the same vertex and focus.



This is easily shown as follows:

The equation of the ellipse referred to its major axis and the tangent at its left-hand vertex, as coördinate axes, is (Part I, Art. 112)

$$\frac{(x-a)^2}{a^2}+\frac{y^2}{b^2}=1, \qquad . \quad . \quad . \qquad (1)$$

which may be written in the form

$$y^2=\frac{2b^2}{a}x-\frac{b^2}{a^2}x^2. \qquad . \quad . \quad . \qquad (2)$$

If now the fixed distance OF be represented by p, then

$$p=OF=OC-FC=a-\sqrt{a^2-b^2},$$

whence

$$b^2=2ap-p^2;$$

therefore

$$\frac{2b^2}{a}=4p-\frac{2p^2}{a}, \quad \text{and} \quad \frac{b^2}{a^2}=\frac{2p}{a}-\frac{p^2}{a^2}.$$

---

* This fact is of importance in astronomy in connection with the behavior of comets.

Substituting these values in equation (2) it becomes

$$y^2 = \left(4p - \frac{2p^2}{a}\right)x - \left(\frac{2p}{a} - \frac{p^2}{a^2}\right)x^2, \quad . \quad . \quad . \quad (3)$$

and the limit of this equation as a approaches ∞, p remaining constant, is

$$y^2 = 4px; \quad . \quad . \quad . \quad (4)$$

which is the equation of a parabola, and the proposition is proved.

In the same way it may be shown that the parabola is the limit to which an hyperbola approaches when its center moves away to infinity, a vertex and the corresponding focus remaining fixed in position (cf. also Note D).

NOTE F

Confocal conics. — Two conics having the same foci, F_1 and F_2, are called **confocal conics.** Since the transverse axis of a conic passes through the foci and its conjugate axis is perpendicular to, and bisects, the line joining the foci, therefore confocal conics are also *coaxial*,* *i.e.*, they have their axes in the same lines. If the equation of any one of such a system of conics is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad . \quad . \quad . \quad (1)$$

and if λ is an arbitrary parameter, then the equation

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1 \quad . \quad . \quad . \quad (2)$$

will represent any conic of the system. For, a and b being constant, and $a > b$, equation (2) represents ellipses for all values of λ between ∞ and $-b^2$, hyperbolas for all values of λ between $-b^2$ and $-a^2$, and imaginary loci when $\lambda < -a^2$; moreover, the distance from the center O to either focus for each of these curves is

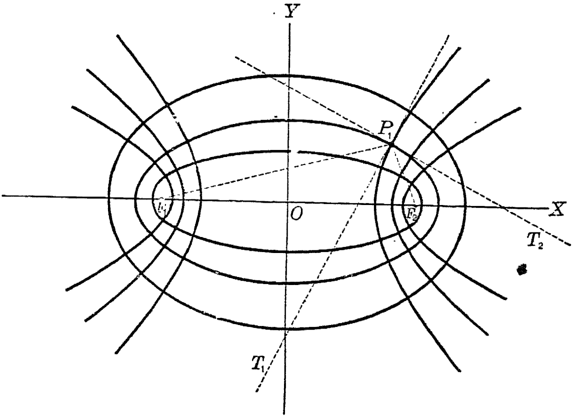
$$\sqrt{(a^2 + \lambda) - (b^2 + \lambda)},$$

which equals $\sqrt{a^2 - b^2}$, and is therefore constant.

The individual curves of the system represented by equation (3) are obtained by giving particular values to λ, each value of λ determining one and but one conic. If any one of these conics is chosen as the fundamental conic, and represented by equation (1), then each of the other conics of the system may be designated by its appropriate value of λ.

---

* Coaxial conics are, however, not necessarily confocal.



Through any assigned point, $P_1 \equiv (x_1, y_1)$, of the plane, there passes one ellipse and one hyperbola of the system represented by equation (2). For substituting the coördinates x_1 and y_1 of P_1 in equation (2), it gives the quadratic equation

$$\frac{x_1^2}{a^2+\lambda} + \frac{y_1^2}{b^2+\lambda} = 1, \quad . \quad . \quad . \quad (3)$$

for the determination of λ. Equation (3) gives two values of λ, hence two conics of this confocal system pass through P_1. That one of these is an ellipse and the other an hyperbola is shown as follows: the quadratic function in λ

$$\frac{x_1^2}{a^2+\lambda} + \frac{y_1^2}{b^2+\lambda} - 1$$

is negative when $\lambda = +\infty$, and, as λ decreases from $+\infty$ to $-\infty$, this function becomes positive just before $\lambda = -b^2$, negative again just after $\lambda = -b^2$, and positive again just before $\lambda = -a^2$; hence, of the two roots of equation (3), one lies between $-b^2$ and ∞, and the other between $-a^2$ and $-b^2$; and therefore of the two confocal conics which pass through P_1, one is an ellipse and the other an hyperbola. Moreover, the two confocal conics which pass through any given point, as $P_1 \equiv (x_1, y_1)$, of the plane intersect at right angles. This is easily seen geometrically thus: connect P_1 with the foci F_1 and F_2, then the tangent P_1T_1 to the hyperbola through P_1 bisects the interior angle between F_1P_1 and F_2P_1, while the tangent P_1T_2 to the ellipse through this same point bisects the external angle formed by these two lines (cf. Part I, Arts. 148, 163); these tangents are therefore at right angles, hence (cf. Part I, Art. 100) the conics intersect at right angles.

This fact could also have been readily proved analytically by comparing the equations of the two tangents.

Remark 1. It is easily seen that as λ approaches $-b^2$ from the *positive* side, the ellipses represented by equation (2) grow more and more flat (because the length of the semi-minor axis $\sqrt{b^2+\lambda}$ approaches 0), approaching, as a limit, the segment F_1F_2 of the indefinite straight line through the foci. On the other hand, if λ approaches $-b^2$ from *below*, then the hyperbolas grow more and more flat, approaching, as a limit, the other two parts of this line. Again, if λ approaches $-a^2$ from *above*, the hyperbolas approach the y-axis as a limit.

Remark 2. Since through every point of a plane there passes one ellipse and one hyperbola of the confocal system represented by equation (2), and but one of each, therefore the two values of λ which determine these two curves may be regarded as the coördinates of this point; they are known as the **elliptic coördinates** of the point. If the rectangular coördinates of a point are known, the elliptic coördinates are easily found by means of equation (2).

E.g., let $P_1 \equiv (x_1, y_1)$ be the point in question, then the elliptic coördinates of P_1 are the two values of λ, which are the roots of equation (3). So, too, if the elliptic coördinates are given, the Cartesian coördinates can be found.

Remark 3. The above observations concerning confocal conics are easily extended to **confocal quadrics**, *i.e.*, to quadric surfaces whose principal sections are confocal conics. They are represented by the equation

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1.$$

13857
172